Schemes by Patching

MTH437 — Introduction to Schemes

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11th October 2021

Recall

We have shown how a scheme can be seen as functor **CRing** to **Set** which satisfies the Zariski Sheaf condition.

The exercises given recently also motivate the idea that sheaves can be thought of as spaces.

We have also motivated how the sheaf condition can be seen as "patching open sets".

As those who have studied manifolds are aware, geometric spaces can be obtained by patching together open sets of a certain type.

In this lecture we will provide two examples to show how $\mathbb{Z}$-affine schemes can be patched to give other schemes.

$\mathbb{A}^2 \setminus \{(0,0)\}$ as a union?

Recall that the quasi-affine scheme $A(x_1, x_2; ; x_1, x_2)$ represents $U = \mathbb{A}^2 \setminus \{(0,0)\}$.

- $U_1 = \mathbb{A}(x_1, x_2, u_1; u_1x_1 - 1)$ represents the subscheme of $\mathbb{A}^2$ where x_1 is "non-zero" (i.e. a unit).
- $U_2 = \mathbb{A}(x_1, x_2, u_2; u_2x_2 - 1)$ represents the subscheme of $\mathbb{A}^2$ where x_2 is non-zero.
- The intersection of U_1 and U_2 in $\mathbb{A}^2$ is represented by the scheme $U_{1,2} = \mathbb{A}(x_1, x_2, u_1, u_2; u_1x_1 - 1, u_2x_2 - 1)$.

Is there some way in which we can obtain the above $\mathbb{Z}$-quasi affine scheme U via "patching U_1 and U_2 along $U_{1,2}$"?

Recall, that a point in $U(R)$ is a pair (r_1, r_2) in R^2 such that $\langle r_1, r_2 \rangle = R$.

Now $(2, 3) \in (\mathbb{A}^2 \setminus \{(0, 0)\})(\mathbb{Z})$, since $\langle 2, 3 \rangle = \mathbb{Z}$.

However $(2, 3)$ is neither in $U_1(\mathbb{Z})$, nor in $U_2(\mathbb{Z})$!

Note that $(2, 3)$ *is* in $U_1(\mathbb{Z}_2)$ *and* in $U_2(\mathbb{Z}_3)$.

Moreover, both these points become the "same" point in $U_{1,2}(\mathbb{Z}_6)$.

We would like such data to be considered as a point in $U(\mathbb{Z})$.

Clarification: Note that we are using the convention $\mathbb{Z}_n = \mathbb{Z}[T]/\langle nT - 1 \rangle$.

This suggests that we consider the following data:

- Elements u_1, u_2 in R such that $\langle u_1, u_2 \rangle = R$;
- for $i = 1, 2$ a point p_i in $U_i(R_{u_i})$;
- the condition that p_1 and p_2 correspond to a point $q_{1,2}$ in $U_{1,2}(R_{u_1 u_2})$.

Such data should be considered as a point in the "union of U_1 and U_2 joined along $U_{1,2}$".

Let us see that this data indeed gives us an R-point of the quasi-affine scheme U.

Warning: Every point in $U(R)$ *need not* be obtained this way! For example, a point of $U_1(R)$ where the second co-ordinate is 0!

All that is being said is that such data *does* give a point in $U(R)$. This was an **error** in an earlier version of these slides.

Suppose $p_i = (p_{i,1}, p_{i,2})$. There is an integer n large enough so that:

- $u_i^n p_{i,j}$ is in R; this is what it means for $p_{i,j}$ to be in R_{u_i}.
- $(u_1 u_2)^n p_{1,j} - (u_1 u_2)^n p_{2,j}) = 0$ in R; this is what it means for $p_{1,j} = p_{2,j}$ in $R_{u_1 u_2}$.

Since $\langle u_1, u_2 \rangle = R$, we can choose x_1, x_2 in R so that the following identity holds in R:

$$u_1^n x_1 + u_2^n x_2 = 1$$

Now consider the point in R^2 given by

$$(a_1, a_2) = (u_1^n x_1 p_{1,1} + u_2^n x_2 p_{2,1} \ , \quad u_1^n x_1 p_{1,2} + u_2^n x_2 p_{2,2})$$

We check that $u_i^n a_j = u_i^n p_{i,j}$ for $i = 1, 2$ and $j = 1, 2$. This shows that (a_1, a_2) gives the same point as p_i in $R_{u_i}^2$ for $i = 1, 2$.

Now, p_i is in $U_i(R_{u_i})$ so that $p_{i,i}$ is a unit in R_{u_i}. It follows that we can find y_i in R so that, by choosing n large enough $u_i^n p_{i,i} y_i = 1$.

It follows that $u_i^n a_i y_i = u_i^n p_{i,i} y_i = 1$.

We now check that

$$(u_1^{2n} x_1 y_1) a_1 + (u_2^{2n} x_2 y_2) a_2 = u_1^n x_1 + u_2^n x_2 = 1$$

which shows that $\langle a_1, a_2 \rangle = R$.

Thus (a_1, a_2) is in $U(R)$.

$\mathbb{P}^1$ as a "union"

Recall the notion of $\mathbb{P}^1(k)$ for a field k.

We can think of $\mathbb{P}^1(k)$ as the union of $V_1(k) = k$ and $V_2(k) = k$ along the overlap $V_{1,2}(k) = k \setminus \{0\}$ which is included in $V_1(k)$ by $t \mapsto t$ and in $V_2(k)$ by $t \mapsto t^{-1}$.

This suggests that we consider:

- two $\mathbb{Z}$-affine schemes $V_1 \simeq \mathbb{A}^1$ and $V_2 \simeq \mathbb{A}^1$,
- patched along $V_{1,2} = A(x_1, x_2; x_1 x_2 - 1)$, where
- $V_{1,2}$ is included V_1 by keeping x_1 and dropping x_2, and
- $V_{1,2}$ is included V_2 by keeping x_2 and dropping x_1.

The resulting "union" should be thought of as $\mathbb{P}^1$.

In terms of the previous description, an R-point p in $\mathbb{P}^1(R)$ *can* be obtained from:

- a pair u_1, u_2 of elements of R such that $\langle u_1, u_2 \rangle = R$,
- for each $i = 1, 2$ a point $p_i \in V_i(R_{u_i}) = R_{u_i}$,
- when considered as points in $V_i(R_{u_1 u_2})$ these points both come from the *same* point $p_{1,2} = (q_1, q_2)$ in $V_{1,2}(R_{u_1 u_2})$.

We see that this means that:

- $q_1 q_2 = 1$ in $R_{u_1 u_2}$
- $p_1 = q_1$ in $R_{u_1 u_2}$
- $p_2 = q_2$ in $R_{u_1 u_2}$.

In other words, the image p_1 under the map $R_{u_1} \to R_{u_1 u_2}$ and the image of p_2 under the map $R_{u_2} \to R_{u_1 u_2}$ satisfy $p_1 p_2 = 1$.

Warning: The point $(1, 0)$ of $V_1(R)$ gives a point of $\mathbb{P}^1(R)$ which is *not* of the above form. As mentioned earlier, there was a **wrong** impression given in the previous slides and lecture that *every* point of $\mathbb{P}^1(R)$ has the above form.

Note that $2/3$ is a point in $V_1(\mathbb{Z}_3)$ and $3/2$ is a point in $V_2(\mathbb{Z}_2)$ such that they correspond to the same point $(2/3, 3/2)$ in $V_{1,2}(\mathbb{Z}_6)$.

Hence, we get a point in $\mathbb{P}^1(\mathbb{Z})$ associated with this data.

It is conventional to denote this point as $(2 : 3)$ as we did earlier in $\mathbb{P}^1(\mathbb{Q})$.

However, not all R-points of $\mathbb{P}^1$ can be written in the form $(a : b)$ where $\langle a, b\rangle = R$.

This has to do with the fact that the ring R need not be a unique factorisation domain.

Consider the ring $R = \mathbb{Z}[x, y, z]/\langle xy - z(1-z)\rangle$.

The elements x/z in R_z and $y/(1-z)$ in R_{1-z} become *inverses* of each other in $R_{z(1-z)}$. Further, it is clear that $\langle z, 1-z\rangle = R$.

So we take

- $p_1 = x/z$ in $V_1(R_z)$ and $p_2 = y/(1-z)$ in $V_2(R_{1-z})$
- $(q_1, q_2) = (x/z, y/(1-z))$ in $V_{1,2}(R_{z(1-z)})$.

Thus we get an R-point p in $\mathbb{P}^1(R)$.

To write p in the form $(a : b)$ in $\mathbb{P}^1(R)$ where $\langle a, b\rangle = R$ we would need $(a : b) = (x : z)$ over $\mathbb{P}^1(R_z)$.

However, no ideal of the form $\langle rx, rz\rangle$ for r in the quotient field of R is a principal ideal.

Conclusion

We have seen two examples of the following kind:

- There are $\mathbb{Z}$-affine schemes U_i for $i = 1, 2$.
- There is an $\mathbb{Z}$-affine scheme $U_{1,2}$.
- The scheme $U_{1,2}$ can be seen as an affine open subscheme of U_1 and U_2.

Note that an affine open subscheme Z_f of a $\mathbb{Z}$-affine scheme Z is given as the locus where a function $f : Z \to \mathbb{A}^1$ is a unit.

If $Z = A(x_1, \dots, x_p; f_1, \dots, f_q)$, then $Z_f = A(x_1, \dots, x_p, u; f_1, \dots, f_q, uf - 1)$ is a subscheme of Z by the morphism that "forgets" u.

In the above context, we showed to be create points of a scheme X that represents "the union of U_1 and U_2 patched along $U_{1,2}$".

In the next lecture we will extend this to more open sets.