Zariski Sheaf functors

We have seen that $\mathbb{Z}$-Affine schemes can be represented as functors **CRing** to **Set** with morphisms represented by natural transformations.

We have also seen that these functors satisfy the following:

Zariski Sheaf property of F: Given a commutative ring R and elements $u_1, \ldots, u_k$ which generate the unit ideal. Given $h_i \in F(R_{u_i})$ for $i = 1, \ldots, k$ such that the images of h_i and h_j in $F(R_{u_i u_j})$ are the same, there is a unique $h \in F(R)$ so that h_i is its image in $F(R_{u_i})$.

Note that images are to be considered under the set maps $F(R) \to F(R_{u_i})$ and $F(R_{u_i}) \to F(R_{u_i u_j})$ induced by the natural ring homomorphisms $R \to R_{u_i}$ and $R_{u_i} \to R_{u_i u_j}$ under the functor F.

We now provide some discussion and examples to justify:

- The notion of schemes needs to be extended by including more functors **CRing** to **Set**.
- We should limit our attention to functors that satisfy the above Zariski sheaf condition.

Geometric Interpretation

The $\mathbb{Z}$-affine scheme $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$ is interpreted as the locus in affine p-space $\mathbb{A}^p$ defined by the vanishing of $f_1, \ldots, f_q$.

In particular, $A(x_1, \ldots, x_p;)$ (a scheme with no equations in p variables) is interpreted as the affine p-space $\mathbb{A}^p$. Note that as a functor, we have $\mathbb{A}^p(R) = R^p$ as expected.

The ring $\mathcal{O}(X)$ associated with a scheme X can be seen as the ring of functions on X or equivalently, morphisms $X \to \mathbb{A}^1$.

In particular, $\mathcal{O}(\mathbb{A}^p) = \mathbb{Z}[x_1, \ldots, x_p]$ is the ring of functions on $\mathbb{A}^p$.

The geometric intuition is that the locus of zeros of functions is *closed*. Moreover, we note that if X is as above then there is a canonical morphism $X \to \mathbb{A}^p$ such that $X(R) \to \mathbb{A}^p = R^p$ makes $X(R)$ into a *subset* of R^p.

Subfunctor: Given a natural transformation $\eta : F \to G$ such that $\eta(R) : F(R) \to G(R)$ makes $F(R)$ into a subset $G(R)$, we say that this makes F into a *subfunctor* of G.

In particular, if F and G are schemes then we will say that F is a *subscheme* of G.

In terms of this terminology we can say that X is a *closed subscheme* of $\mathbb{A}^p$. In other words, what we have been calling $\mathbb{Z}$-affine schemes can also be called closed subschemes of $\mathbb{A}^p$.

Set-theoretic constructions (Simple cases)

Some set-theoretic constructions have natural geometric meaning so we try to give functorial analogues.

Product

Given sets U and V we can form the product $U \times V$ which consists of pairs (u, v) with u from U and v from V.

Given functors F and G from **CRing** to **Set** it is not difficult to see that there is a natural functor $F \times G$ as follows:

- For a ring R, we define $(F \times G)(R) = F(R) \times G(R)$
- For a ring homomorphism $f : R \to S$, we define $(F \times G)(f) = F(f) \times G(f)$.

In particular, we can apply this to the $\mathbb{Z}$-affine schemes $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$ and $Y = A(y_1, \ldots, y_r; g_1, \ldots, g_s)$. We note that $X_. \times Y_.$ is the functor $Z_.$ where

$$Z = A(x_1, \ldots, x_p, y_1, \ldots, y_r; f_1, \ldots, f_q, g_1, \ldots, g_s)$$

Here, we have used the fact that x_i and y_j are *dummy* variables to merge them without overlap!

In fact, we note that $\mathcal{O}(Z) = \mathcal{O}(X) \otimes \mathcal{O}(Y)$ where the latter is the tensor product of the two abelian groups which has a natural ring structure as well.

Diagonal

Given a set U, we can consider it as a subset $\Delta : U \to U \times U$ via the map that sends u to the pair (u, u).

Similarly, given a functor F from **CRing** to **Set**, we can produce a natural transformation $\Delta : F \to F \times F$ that exhibits F as a subfunctor of $F \times F$.

Applying this to a $\mathbb{Z}$-affine scheme $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$ we note that Δ exhibits X as the subscheme of $X \times X$ defined by

$$\Delta_X = A\left(x_1, \ldots, x_p, y_1, \ldots, y_p; \right. \\ \left. f_1(\mathbf{x}), \ldots, f_q(\mathbf{x}), f_1(\mathbf{y}), \ldots, f_q(\mathbf{y}), x_1 - y_1, \ldots, x_p - y_p\right)$$

Intersection

Given subsets U and V in a set W, we have the intersection $U \cap V$ as a subset of W.

Similarly, given subfunctors F and G of a functor H from **CRing** to **Set**, we have the intersection $F \cap G$ as a subfunctor of H.

Since every $\mathbb{Z}$-affine scheme is a subscheme of $\mathbb{A}^p$ for some p, it is enough to consider the intersection of two subschemes $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$ and

$Y = A(x_1, \ldots, x_p; g_1, \ldots, g_r)$ in $\mathbb{A}^p$. This is the subscheme $X \cap Y$ defined by

$$X \cap Y = A(x_1, \ldots, x_p; f_1, \ldots, f_q, g_1, \ldots, g_s)$$

Inverse-image

Given a map $f : U \to V$ and a subset W of V, we have a subset $f^{-1}(W)$ of U called the inverse image of W under f.

$$f^{-1}(W) = \{x \in U | f(x) \in W\}$$

Similarly, given a natural transformation $\eta : F \to G$ and a subfunctor H of G, where all of these are functors from **CRing** to **Set**, we have a subfunctor $\eta^{-1}(H)$ of F.

Since every $\mathbb{Z}$-affine scheme is a subscheme of $\mathbb{A}^p$ for some p, it is enough to consider the inverse image of a subscheme Y of $\mathbb{A}^p$ under a morphism $h : X \to \mathbb{A}^p$.

Suppose that $X = A(x_1, \ldots, x_r; f_1, \ldots, f_s)$ and $Y = A(y_1, \ldots, y_p; g_1, \ldots, g_q)$.

Since h is given by a ring homomorphism $\mathbb{Z}[y_1, \ldots, y_p] \to \mathcal{O}(X)$ it is given by polynomials $h_1, \ldots, h_p$ in the variables $x_1, \ldots, x_r$ such that $f_i(h_1, \ldots, h_r) = 0$ for all $i = 1, \ldots, s$.

We then see that $h^{-1}(Y) = W$ is defined by

$$W = h^{-1}(Y) = A(x_1, \ldots, x_r; f_1, \ldots, f_s, g_1(\mathbf{h}), \ldots, g_s(\mathbf{h}))$$

where

$$g_i(\mathbf{h}) = g_i\left(h_1\left(x_1, \ldots, x_r\right), \ldots, h_r\left(x_1 \ldots, x_r\right)\right)$$

is a polynomial in the variables $x_1, \ldots, x_r$.

Fibre-product

All of the above constructions are related to the notion of "Fibre-product". Given set maps $f : U \to W$ and $g : V \to W$, the *fibre-product* $T = U \times_W V$ is defined by

$$T = U \times_W V = \{(u, v) \in U \times V | f(u) = g(v)\}$$

we note that it is a subset of $U \times V$. In fact, there is a natural map $U \times V \to W \times W$ and the fibre-product is the inverse image of the diagonal Δ_W in $W \times W$.

Similarly, it is not difficult to check that if $U \to W$ and $V \to W$ are subsets of W, then $U \times_W V = U \cap W$.

Disjoint Union

Given two sets U and V, we can form the disjoint union $U \sqcup V$. Similarly, given two functors F and G from **CRing** to **Set** we can form $F \sqcup G$ such that

$$(F \sqcup G)(R) = F(R) \sqcup G(R)$$

However, this functor *does not* represent our geometric intuition when F and G are geometric functors as we shall see below.

Suppose that $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$ and $Y = A(y_1, \ldots, y_r; g_1, \ldots, g_s)$. Let us now examine the question of what $X \sqcup Y$ could be.

Recall that $R = \{0\}$ represents the empty space . There is only *one* map from the empty space to any space. Thus $(X \sqcup Y)(R)$ should be a singleton! However $X_.(R) \sqcup Y_.(R)$ is the disjoint union of two singletons and so has 2 elements.

So $X_. \sqcup Y_.$ is not the "right" choice to represent the geometric disjoint union of X and Y.

The direct sum of rings

Functions on the disjoint union $X \sqcup Y$ of geometric spaces X and Y are *pairs* (a, b) where a is a function on X and b is a function on Y. Moreover, addition and multiplication are "entry-wise".

This suggests that $\mathcal{O}(X \sqcup Y) = \mathcal{O}(X) \oplus \mathcal{O}(Y)$. Note also that $(0, 0)$ and $(1, 1)$ serve as the 0-element and the 1-element respectively. Note that this ring has two idempotents $e_X = (1, 0)$ and $e_Y = (0, 1)$ which satisfy

- $e_X^2 = e_X$ and $e_Y^2 = e_Y$
- $e_X e_Y = 0$ and $e_X + e_Y = 1$

Such a pair of idempotents in a ring is called a *decomposition of identity into a pair of orthogonal idempotents.*

We can check that

$$\mathcal{O}(X) \oplus \mathcal{O}(Y) \cong \frac{\mathbb{Z}[u, x_1, \ldots, x_p, y_1, \ldots, y_r]}{\langle f_1, \ldots, f_q, g_1, \ldots, g_s, u(1-u), ux_1, \ldots, ux_p, (1-u)y_1, \ldots, (1-u)y_r \rangle}$$

Here u and $1 - u$ are give the required pair of idempotents.

In other words, this ring is associated with the $\mathbb{Z}$-affine scheme

$$A\left(u, x_1, \ldots, x_p, y_1, \ldots, y_r; f_1, \ldots, f_q, g_1, \ldots, g_s, u(1-u), ux_1, \ldots, ux_p, (1-u)y_1, \ldots, (1-u)y_r\right)$$

We will now use $X \sqcup Y$ for this affine scheme, but use $\mathcal{O}(X) \oplus \mathcal{O}(Y)$ in place of the above more cumbersome notation (using u) in place of the ring $\mathcal{O}(X \sqcup Y)$.

The question remains why $\mathcal{O}(X) \oplus \mathcal{O}(Y)$ is the "right" choice. So we check that it does the "right" things.

Case where $R = \{0\}$

First of all, let us note that there *is* only one homomorphism from *any* ring to the ring $\{0\}$. Thus, as required, $(X \sqcup Y)(\{0\})$ is a singleton!

Case where R has only trivial idempotents

Now, if R is a ring where the *only* idempotents are 0 and 1 with $1 \neq 0$, then a homomorphism $f : \mathcal{O}(X) \oplus \mathcal{O}(Y) \to R$ has the property that exactly one of the following holds:

- $f(e_X) = 1$ and $f(e_Y) = 0$
- $f(e_X) = 0$ and $f(e_Y) = 1$

It follows that *if* R is a ring with 0 and 1 as the only idempotents, and $1 \neq 0$ then

$$\text{Hom}\left(\mathcal{O}(X) \oplus \mathcal{O}(Y), R\right) = \text{Hom}\left(\mathcal{O}(X), R\right) \sqcup \text{Hom}\left(\mathcal{O}(Y), R\right)$$

Here the first term on the right is identified with maps that are 0 on $\mathcal{O}(Y)$ and the second term on the right is identified with maps that are 0 on $\mathcal{O}(X)$. So in this case,

$$X(R) \sqcup Y(R) = (X \sqcup Y)(R)$$

Exercise: How did we use $1 \neq 0$?

Case where R has non-trivial idempotents

When R *does* have a non-trivial idempotent e_1 (i.e. e_1 and $e_2 = 1 - e_1$ are *both* non-zero), the situation becomes more complicated.

Note that even in this case, the previous calculations show that

$$X(R) \sqcup Y(R) \ \subset (X \sqcup Y)(R)$$

In addition to homomorphisms on the left-hand side, we can have a ring homomorphism $f : \mathcal{O}(X) \oplus \mathcal{O}(Y) \to R$ with $f(e_X) = e_1$ and $f(e_Y) = e_2$. We can also have a ring homomorphism $f : \mathcal{O}(X) \oplus \mathcal{O}(Y) \to R$ with $f(e_X) = e_2$ and $f(e_Y) = e_1$.

Note that e_1 and e_2 give a decomposition of identity into a pair of orthogonal idempotents in the ring R. It follows that $R_{e_1} = Re_1$ and $R_{e_2} = Re_2$. Note also that $R = Re_1 \oplus Re_2$ and $R_{e_1 e_2} = \{0\}$.

A homomorphism $f : \mathcal{O}(X) \oplus \mathcal{O}(Y) \to R$ such that $f(e_X) = e_1$ gives rise to elements $f_1 \in \text{Hom}(\mathcal{O}(X), Re_1)$ and $f_2 \in \text{Hom}(\mathcal{O}(Y), Re_2)$. So we have

$$f_1 \in X(R_{e_1}) \subset (X \sqcup Y)(R_{e_1})$$
$$f_2 \in Y(R_{e_2}) \subset (X \sqcup Y)(R_{e_2})$$

Moreover, their images in $(X \sqcup Y)(R_{e_1 e_2})$ are the same since this is a *singleton*.

The existence of an element f in $(X \sqcup Y)(R)$ in this case is an application of the *sheaf condition*! This shows us the importance of the sheaf condition.

Exercise: Show that disjoint union $X_\cdot \coprod Y_\cdot$ *as functors* does not satisfy the sheaf condition.

Complement

Given a subset V of a set U, we can form the complement $U \setminus V$.

However, if G is a subfunctor F of functors **CRing** to **Set**, then we do not have a functor that associates $F(R) \setminus G(R)$ to the ring R for every ring R. The reason is that for some ring homomorphism $f : R \to S$, some element of $F(R) \setminus G(R)$ may have image in $G(S)$ under $F(f)$.

For example, consider a $\mathbb{Z}$-affine scheme $Y = A(x; x)$ as a subscheme of $\mathbb{A}^1$ and the ring homomorphism $f : \mathbb{Z} \to \mathbb{Z}/\langle 5 \rangle$. We have the $\mathbb{Z}$-point of $\mathbb{A}^1$ given by the homomorphism $\mathbb{Z}[x] \to \mathbb{Z}$ that maps x to 5 whose image under f is in $Y(\mathbb{Z}/\langle 5 \rangle)$.

More generally, if we want to have a notion of the complement of $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$ in $\mathbb{A}^p$, we have to ensure that an R-point in the "complement of X" should go to an S-point in the "complement of X" for *every* ring homomorphism $f : R \to S$.

Now an R-point of $\mathbb{A}^p$ can be seen as a ring homomorphism $\mathbf{a} : \mathbb{Z}[x_1, \ldots, x_p] \to R$. Saying that it is in the complement of $X(R)$ means that the image ideal $I = \mathbf{a}\langle f_1, \ldots, f_q \rangle R$ is *not* the zero ideal in R.

The above condition, means we want the ideal I in R such that its image $f(I)S$ under every homomorphism $f : R \to S$ is a non-zero ideal. Since we can always take $S = R/I$, this appears to be problematic!

Now, we already decided that maps from to any space is a singleton whereas $\mathbb{A}^p(\{0\}) - X(\{0\})$ is empty! Thus, we can allow the the image of I to be $\{0\}$ in the case of $f : R \to \{0\}$.

So one way to state the above condition is that $I = R$. Now, the image ideal $f(I)S$ under any homomorphism $f : R \to S$ also satisfies $f(I)S = S$.

Quasi-affine $\mathbb{Z}$-scheme

A quasi-affine $\mathbb{Z}$-scheme is denoted $A(x_1, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$.

We conceptually think of this as the locus of points in $\mathbb{A}^p$ which satisfy the equations $f_i = 0$ for $i - 1, \ldots, q$ and $\langle g_1, \ldots, g_r \rangle$ "do not all vanish".

To the quasi-affine scheme $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$ we associate a functor of points $X_.$ from **CRing** to **Set**. This associates to a ring R, the set

$$X_.(R) = \{\mathbf{a} = (a_1, \ldots, a_p) \mid \langle f_1(\mathbf{a}), \ldots, f_q(\mathbf{a}) \rangle_R = \langle 0 \rangle_R \\ \text{and } \langle g_1(\mathbf{a}), \ldots, g_r(\mathbf{a}) \rangle_R = R\}$$

One special case is when $r = 1$. In that case, the requirement that $\langle g_1(\mathbf{a}) \rangle = R$ is the same as the requirement that $g_1(\mathbf{a})$ is a unit in R. Hence, we see that

$$A(x_1, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r) = A(x_1, \ldots, x_p, u; f_1, \ldots, f_q, ug_1 - 1;)$$

which is a $\mathbb{Z}$-affine scheme.

In general, given $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$, there *need not* be a $\mathbb{Z}$-affine scheme Y and a natural transformation $X \to Y$ which is a bijection on R-points for all R. For example, we can see this for $\mathbb{A}^2 \setminus \{(0,0)\}$ as we shall see.