Solutions to Quiz 3

Consider the category **Grp** whose objects are groups and morphisms are group homomorphisms.

Which of the following associations can be made into functors?

Q1

Take a group G to the abelian group $G^{ab} = G/[G,G]$ where $[G,G]$ is the subgroup generated by commutators.

We note that if $f : G \to A$ is a group homomorphism and A is abelian then $f([G,G]) = e$. Hence, f factors as $G \to G^{ab} \to A$.

It follows that if $f : G \to H$ is a group homomorphism, the composite map $G \to H \to H^{ab}$ factors as $f^{ab} : G^{ab} \to H^{ab}$.

One then checks that this gives a functor.

Q2

For a fixed field k, take a group G to the set $\mathcal{C}(G)$ of conjugacy classes in G.

A group A is group if and only every conjugacy class has one element.

Now, if $f : A \to G$ is a group homomorphism, then the image of a conjugacy class need not be a conjugacy class if G is not abelian.

For example, if G is a group with elements a and b such that $ab \neq ba$, then consider the natural homomorphism $\mathbb{Z} \to G$ given by $1 \mapsto a$, The set $\{a\}$ is the image of a conjugacy class in $\mathcal{C}(\mathbb{Z})$. However, the conjugacy class of a at least contains the subset $\{a, b^{-1}ab\}$ which has two elements.

Q3

Take a group G to the set $\mathrm{Hom}(\mathbb{Z}/(2), G)$ of homomorphisms from the group with 2 elements.

Given a group homomorphism $f : G \to H$ and an element a in $\mathrm{Hom}(\mathbb{Z}/(2), G)$, we see that $f \circ a$ is an element of $\mathrm{Hom}(\mathbb{Z}/(2), H)$.

In fact, this is the functor $(\mathbb{Z}/(2))^{\cdot}$ introduced in class.

Q4

Take a group G to the group $G \times G$ which is the product of G with itself.

Given a group homomorphism $f : G \to H$, the map $f \times f : G \times G \to H \times H$ is also a group homomorphism.

One checks that this has the properties of a functor.

Q5

Take group G to the set G_5 of elements of order 5 in G.

(The following was pointed out by Biplob!)

Given a group homomorphism $G \to H$ where G has elements of order 5 and H has no elements of order 5, the set H_5 is empty. Hence, there is no natural map $G_5 \to H_5$ since there are no set maps from a non-empty set to the empty set.

If we replace the above definition by taking G_5 as elements of order *dividing* 5, then we can check that there is a functor. In fact, this is $(\mathbb{Z}/(5))^{\cdot}$.

Q6

Take a group G to set $\mathcal{N}(G)$ of normal subgroups of G

For reasons similar to those given for conjugacy classes one can check that the image of a normal subgroup need not be normal.

If we are looking for a *contravariant* functor, then it is possible. This is a functor from $\mathbf{Grp}^{op}$ to $\mathbf{Set}$.

Given a group homomorphism $f : G \to H$ and a normal subgroup N of H, we note that $f^{-1}(N)$ is the kernel of the composite homomorphism $G \to H \to H/N$; hence it is a normal subgroup. This gives a map $f^{-1} : \mathcal{N}(H) \to \mathcal{N}(G)$.