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You have defined Z-affine schemes in terms of functors, as you mentioned in the previous class this is a different point of view compared to the "classical" definition of schemes as certain locally ringed spaces. You mentioned this modern point of view is more intuitive. Can you elaborate on this?

I have seen schemes from the classical point of view, I want to get an idea where these two standpoints overlap and when they differ.

$$X = A(x_1, \dots, x_p; f_1, \dots, f_q) \rightsquigarrow \mathcal{O}(X)$$

$$\mathcal{O}(X) = \mathbb{Z}[x_1, \dots, x_p] / \langle f_1, \dots, f_q \rangle$$

$$\mathfrak{X} = \operatorname{Spec} \mathcal{O}(X)$$

$$\mathfrak{X} \rightsquigarrow X. \qquad \underline{\text{CRing}} \rightsquigarrow \underline{\text{Set}}$$

$$R \rightsquigarrow X(R) = \operatorname{Hom}(\mathcal{O}(X), R)$$

$$\operatorname{Spec} R \to \operatorname{Spec} S \longleftrightarrow S \to R$$

$$X(R) = \operatorname{Mor}(\operatorname{Spec} R, \mathfrak{X})$$

$$\mathcal{O}(X) = \mathfrak{X}$$

($\operatorname{Spec} R$ is a topological space + ...)

R-points are not points of the topological space

pt. of $\mathfrak{X}$

$$F \not\to \mathfrak{X}$$

for every R

$$F(R) = \underline{\text{pt. of } X(R)}$$

$$\mathbb{Z}[X]/(X, 5) \qquad \mathbb{Z}/(5)$$

$F(R)$ is a singleton or empty.

7. $F = A(x_1, \dots, x_n; f_1, \dots, f_n)$

$\underbrace{O}(X) \to R$

either there is one map ~~of~~ or not maps.

✓ $\mathbb{Z}_1 \to R$ there is a unique map

$1 \mapsto 1$

✗ $\mathbb{Z}[i] \to \mathbb{Z}[i]$

$i \mapsto i$

$i \mapsto -i$

✗ $\mathbb{Z}[x] \to \mathbb{Z}[x]$

$X \mapsto p(x)$

$S \to R$ either there is one map or no maps

$\{0\} \to R$ $\quad \dfrac{\mathbb{Z}[x_1,\dots,x_n]}{\langle 1 \rangle} = \{0\}$ ✓

$0 \to 0$

$\| \quad \|$

$1 \to 1$

$\mathbb{Z} \to \mathbb{F}_p \to R$

p prime

✓ $1 \mapsto 1$

$p \mapsto 0$

R is an $\mathbb{F}_p$ algebra

~~Sop~~

$(\mathbb{I}) \to R$

$\mathbb{Z}_1[x]/(aX-1)$ $\quad a \in \mathbb{Z}$

$a \quad 1$

$\downarrow \quad \downarrow$

$a \quad 1$

R

R = domain $\mathbb{Z}[x]$

$Q(R) = \mathbb{Q}(x)$

$\mathbb{Q}(x) \to \mathbb{R}$

$\mathbb{Z}[x] \to R$

many maps.

$\mathbb{Z}, \mathbb{F}_p, \{0\}, \mathbb{Z}[X]/(nX-1)$

R

$\frac{a}{b} = \frac{c}{d}$ $\quad (ad-bc) \cdot u = 0$

$\frac{ua}{ub} = \frac{a}{b}$

$R_u \to Q(R)$ = invert non-zero divisors

u non-zero divisor

= Ring of fractions of R

$\mathrm{Spec}(R)_{top} = \{$ prime ideals in $R\}$

Zariski top.

"Minimal" Definition

"Locally Ringed Space"

$k \rightsquigarrow \mathbb{Z}$

k- Schemes (finite type)

k- field

$X = A(x_1,..,x_p ; f_1,..,f_q)$

$f_i \in k[x_1,..,x_p]$

$k =$ algebraically closed
characteristic 0
& maybe even $\mathbb{C}$

<u>Algebraic Sets</u>

$X(\mathbb{C})$ = topol subspace of $\mathbb{C}^p$

<u>Zariski topology</u>

Closed set are given by
"Closed Subschemes"

~~$X(\mathbb{C})$~~ $\{ a \in X(\mathbb{C}) \mid f_1(a), .. f_q(a) = 0, \; g_1(a) ..., g_r(a) = 0 \}$

① Arithmetic check the theorem for other fields. ($\overline{\mathbb{F}_p}$ .....)

② Galois actions

③ $X = A(x, y ; x^2 + y^2 - 1)$

$\mathbb{C}$-scheme. $Y_0 = A_{\mathbb{C}}(x, y ; x = 0)$

$Y_1 = A(x, y ; x = 1)$

$\alpha \in \mathbb{C}$ $Y_\alpha = A(x, y ; x = t)$

$X \cap Y_t$ ~~$t \neq \pm 1$~~ $t \neq \pm 1$

there are two points in $X \cap Y_t(\mathbb{C})$
$t \neq \pm 1$ $\Rightarrow$

$X = \mathbb{A}_{\mathbb{C}}(x,y\,;x^2+y^2-1)$

$\alpha\in\mathbb{C}$ $Y_\alpha = \mathbb{A}_{\mathbb{C}}(x,y\,;x-\alpha)$

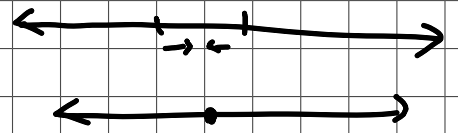
$\underline{X(\mathbb{C})}\cap Y_\alpha(\mathbb{C}) = \{(x,y) \mid y^2 = 1+\alpha^2\}$

$\alpha \neq \pm 1 \Rightarrow 2$ points

$\alpha = \pm 1 \Rightarrow 1$ point

"Principle of continuity"

"Multiple pts"



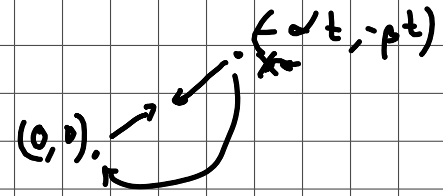
What ring?

(Rings associated with each open set

Ringed space)

"local ring" is attached

←



$\mathbb{C}[x,y]\big/\big((x,y)\cap(x+\alpha t, y+\beta t)\big)$

Rings and nilpotent allow one to capture <u>tangent</u> <u>direction</u>

$$\mathcal{O}(X) = \frac{\mathbb{C}[x,y]}{(x^2+y^2-1)} \qquad \mathcal{O}(Y_\alpha) = \frac{\mathbb{C}[x,y]}{(x-\alpha)} \simeq \mathbb{C}[y]$$

$$\mathcal{O}(X\cap Y_\alpha) = \frac{\mathbb{C}[x,y]}{(x^2+y^2-1,\ x-\alpha)}.$$

$$\simeq \frac{\mathbb{C}[y]}{(y^2-(1-\alpha^2))} \simeq \mathbb{C}\oplus\mathbb{C} \quad \alpha\neq 0$$

$$\simeq \frac{\mathbb{C}[y]}{y^2} \quad \alpha = 0$$

"double point"

$A(x\,;x) \neq A(x\,;x^2)$

$\{0\}$ $\{0\}$

Set-theoretically same complex solution.

→ Keep track of ring