Set-Theoretic Constructions for Schemes

$\mathbb{A}^p = A(x_1,\dots,x_p;\ ) \supset$ Obj. $A(x_1,\dots,x_p;$
$x_1,\dots,x_p)$

$\cup$

$X = A(x_1,\dots,x_p; f_1,\dots,f_q)$

"closed" $\quad \mathbb{A}^p \xrightarrow{f} \mathbb{A}^q \quad a \mapsto (f_1(a),\dots,f_q(a))$

$X = f^{-1}(\{\text{origin}\})$

$\mathbb{A}^p(R) = R^p \supset X(R)$

$Y = A(x_1,\dots,x_p; g_1,\dots,g_s)$

$X(R) \cap Y(R) = \cancel{A(x_1,\dots,x_p, f_1g_1, f_1g_2,\dots} \cancel{f_2g_1, f_qg_s\dots)}$

$\cancel{A(x_1,\dots,x_p, (f_ig_j)}_{i=1, j=1}^{q, s}$

$= A(x_1,\dots,x_p; f_1,\dots,f_q, g_1,\dots,g_s)$

$X \times Y \qquad X = A(x_1,\dots,x_p; f_1\dots f_q)$
$\qquad\qquad Y = A(y_1,\dots,y_r; g_1,\dots,g_s)$
$\parallel$

- $A(x_1,\dots,x_p; y_1,\dots,y_r; f_1,\dots,f_q, g_1,\dots,g_s)$
- $X, Y \subset \mathbb{A}^p \qquad r = p$

$r = p$

$$(X \cap Y) = (X \times Y) \cap \Delta$$

$$\Delta = A(x_1,\dots,x_p, y_1,\dots,y_p;\ (x_i - y_i)_{i=1}^p)$$

$X, Y \subset S \qquad X \times Y \subset S \times S \supset \Delta_S$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \{(s,s)\}$

$X \times Y \cap \Delta = X \cap Y$

$$\begin{array}{ccc} \mathbb{Z}[x_1,\dots,x_p] & & \mathbb{Z}[y_1,\dots,y_r] \\ & \searrow \quad \swarrow & \\ & \mathbb{Z}[x_1,\dots,x_p, y_1,\dots,y_r] & \end{array}$$

Tensor product of rings

R, S rings. $R = \mathbb{Z}[x_1, \dots, x_p]$
$S = \mathbb{Z}[y_1, \dots, y_r]$

$R \otimes S = \mathbb{Z}[x_1, \dots, x_p, y_1, \dots, y_r]$

$\mathbb{Z}[x_1, \dots, x_p]$

$R \longrightarrow R \otimes S$
$a \longmapsto a \otimes 1$

$1 \otimes b$

$S \quad b$

$R \otimes S \qquad \boxed{\sum a_i \otimes b_i}$

$(a+b) \otimes c = a \otimes c + b \otimes c$

$3a \otimes b = (a+a+a) \otimes c$
$= a \otimes c + a \otimes c + a \otimes c = a \otimes 3c$

$n a \otimes b = a \otimes n b \quad n \in \mathbb{Z}$

$\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes \mathcal{O}(Y)$

R, S are comm rings with 1

$R \oplus S \qquad (1,1) = (1,0) + (0,1)$
$1 = e_S + e_R$

$e_S^2 = e_S \quad e_R^2 = e_R \quad e_S \cdot e_R = 0$

Component-wise operations.

$\mathcal{O}(X) \oplus \mathcal{O}(Y) = \mathcal{O}(\, ?? \,)$

$(X \amalg Y)$

$(X \amalg Y)(R) \overset{?}{=} X(R) \amalg Y(R)$

$\mathcal{O}(X) \oplus \mathcal{O}(Y) \longrightarrow R \qquad e_x \cdot e_y = 0$

$X(Re_1) \amalg Y(Re_2) \quad e_x + e_y = 1 \quad e_x^2 = e_x \quad e_y^2 = e_y$

$e_x \longrightarrow e_1 \qquad e_y \to e_2 \quad e_1 + e_2 = 1$

$Re_1 \oplus Re_2 = \{a \in R \mid ae_2 = a\}$

$\{a \in R \mid ae_1 = a\}$

$a \cdot 1 = ae_1 + ae_2 \qquad ae_1 \cdot e_1 = ae_1 \in Re_1$
$ae_2 \cdot e_2 = ae_2 \in Re_2$

① Complements. ② Unions/Disjoint union.

③ Non-disjoint unions

$$X = A(x_1,..,x_p; f_1,...,f_q)$$
$$Y = A(x_1,..,x_p; g_1,..,g_s)$$

$$X \cup Y \;\cancel{X \cap Y} \overset{?}{=} A(x_1,..,x_p; (f_i g_j)_{i=1, j=1}^{p,\ s})$$

$$X \cup X \overset{?}{=} A(x_1,..,x_p; (f_i f_j)_{i=1, j=1}^{p,\ p})$$

$$X = A(x_1,..,x_p; x_1) \quad . \quad \mathbb{Z}[x_1,..,x_p]/\langle x_1\rangle$$
$$\uparrow$$
$$X \cup X \overset{?}{=} A(x_1,..,x_p; x_1^2) \quad \mathbb{Z}[x_1,..,x_p]/\langle x_1^2\rangle$$

Set-Theory not quite correct

+ "Sheaf" property

F Cring $\rightsquigarrow$ Set

F : Cring $\rightsquigarrow$ Set

R, $u_1,...,u_r \in R$ $\langle u_1,..,u_r\rangle_R = R$

$$F(R) \longrightarrow \boxed{F(R_{u_i})} \rightrightarrows F(R_{u_i u_j})$$

is "exact"

$$R \xrightarrow{f_i} R_{u_i} \qquad R_{u_i} \xrightarrow{g_{i,j}} R_{u_i u_j} = R_{u_j u_i}$$

$a \in F(R)$ $\quad F(f_i): F(R) \to F(R_{u_i})$

$$a \longmapsto F(f_i)(a) = a_i$$

- $g_{i,j}(a_i) = g_{j,i}(a_j)$
- given $b_i \in F(R_{u_i})$ s.t. $g_{i,j}(b_i) = g_{j,i}(b_j)$

$$\exists!\, b \in F(R) \text{ s.t. } b_i = F(f_i)(b)$$

$X \amalg Y$ for rings R with idempotent.

$e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 e_2 = 0$

$\langle e_1, e_2\rangle = R$ $\qquad e_1 + e_2 = 1$

$R_{e_1} = R\cdot e_1 \qquad R_{e_2} = R\cdot e_2 \qquad (Re_1)_{e_2} = \{0\}$

$X(Re_1)$, $Y(Re_2)$ $\qquad (X \amalg Y)(R)$

$a_1 \qquad a_2 \qquad a$

$X(R) \amalg Y(R) \qquad X(Re_1) \amalg Y(Re_2)$

$\mathcal{O}(X) \oplus \mathcal{O}(Y) = \mathbb{Z}$

$\downarrow$

is associated with a $\mathbb{Z}$-affine scheme

$(X \amalg Y)$

if R has only $0,1$ as idempotents

$$\mathcal{O}(X) \oplus \mathcal{O}(Y) \longrightarrow R$$

$e_X \quad e_Y \qquad e_X + e_Y = 1$

$e_X \to 0$ or $e_Y \to 0$

$$(X \amalg Y)(R) = X(R) \amalg Y(R)$$

$\uparrow$

However not true when R has other idempotents

$e \in R \quad e \neq 0,1 \qquad e_1 = e\,,\ e_2 = 1-e$

$e(1-e) = 0 = e_1 e_2$

$$(X \amalg Y)(R) \supset X(Re) \amalg Y(R(1-e))$$

$\mathbb{Z}$-affine schemes satisfy sheaf condition.

"Modify set-theoretic expectation using sheaf-theory & patching"