Localisation and Patching

We discuss an important property of the functor of points $X_.$ from **CRing** to **Set** associated with a $\mathbb{Z}$-Affine scheme X.

Universal property of $R[T]$

Given a homomorphism $f : R \to S$ of commutative rings and an element t in S, we obtain a ring homomorphism $g : R[T] \to S$ which sends T to t. Moreover, $g(r) = f(r)$ for every element of R.

In fact, we can identify the set

$$\{h : R[T] \to S : h(r) = f(r)\}$$

with S by identification $h \mapsto h(T)$.

Put differently,

$$\text{Hom}(R[T], S) = \text{Hom}(R, S) \times S$$

Note that $\mathcal{O}$ is a functor $\mathbb{Z}$-Aff to **CRing**.

We also have the "forgetful functor" **CRing** to **Set** that "forgets" the ring structure. It associates a ring to the underlying set and a ring homomorphism to the underlying set map.

For a $\mathbb{Z}$-Affine scheme X, then we have

$$\text{Mor}(X, A(x;)) = \text{Hom}(\mathbb{Z}[x], \mathcal{O}(X)) = \mathcal{O}(X)$$

So $\mathcal{O}$ can be identified with $A(x;)_.$!

Localisation

Given a commutative ring R and an element u in R, we define

$$R_u = R[T]/\langle uT - 1\rangle$$

which is called the *localisation* of R at u.

We note the following properties.

- If u is nilpotent, then $R_u = \{0\}$ is the 0-ring.
- u is invertible in R_u with the image t of T being the multiplicative inverse of u.
- Every element of R_u can be written in the form rt^k (or in other notation r/u^k) for some r in R and k a non-negative integer.
- Elements r_1/u^{k_1} and r_2/u^{k_2} in R_u are equal if there is an integer k such that $u^k(r_1u^{k_2} - r_2u^{k_1}) = 0$ in R.
- In particular, there is a natural ring homomorphism $R \to R_u$ such that its kernel consists of elements annihilated by some power of u.

The ring R_u is called the *localisation* of R at u.

Universal property of localisation

Given a commutative ring homomorphism $f : R \to S$ so that $f(u)$ is invertible in S with inverse t. As seen above, we can use t to extend f to a ring homomorphism $R[T] \to S$ which sends T to t. Note that $uT - 1$ is mapped to $f(u)t - 1 = 0$. So we have a homomorphism $\tilde{f} : R_u \to S$.

Conversely, given a homomorphism $g : R_u \to S$, let f denote the composite $R \to R_u \to S$. Since the image of u in R_u is invertible, so is $f(u)$.

In other words, a ring homomorphism $f : R \to S$ *factors* as $R \to R_u \to S$ if and only if $f(u)$ is invertible in S.

Localisation of Localisation

Given elements u and v in R, we consider the ring R_{uv}. Since uv has an inverse t in the latter ring, we have $tuv = 1$ in R_{uv}. Hence, we see that tu is an inverse of v in this ring as well.

It follows that $R \to R_{uv}$ factors as $R \to R_v \to R_{uv}$.

Similarly, since tv is an inverse of u in R_{uv}, we see that that $R_v \to R_{uv}$ factors as $R_v \to (R_v)_u \to R_{uv}$.

Conversely, if w denotes the image of the inverse of v in R_v under $R_v \to (R_v)_u$, and z denotes the inverse of u in $(R_v)_u$, then wz is an inverse of uv in $(R_v)_u$.

This gives a ring homomorphism $R_{uv} \to (R_v)_u$ which is the inverse of the above homomorphism.

Henceforth, we will treat these isomorphisms as *identity*!

In addition, we will identify the rings R_{u^2} and R_u for similar reasons.

Patching

Given elements $u_1, \ldots, u_k$ in R, we have seen above that there are ring homomorphisms $f_i : R \to R_{u_i}$ and $g_{i,j} : R_{u_i} \to R_{u_i u_j}$, for each i and j.

Suppose $(r_i)_{i=1}^k$ is a collection of elements with $r_i \in R_{u_i}$. We can ask for a condition under which there is an element r in R such that $f_i(r) = r_i$ for $i = 1, \ldots, k$.

Note that if r is in R, then $g_{i,j}(f_i(r)) = g_{j,i}(f_j(r))$ for all i and j. Thus, a *necessary* condition is that

$$g_{i,j}(r_i) - g_{j,i}(r_j) \text{ for } i < j$$

The question is to identify a condition on $(u_1, \ldots, u_k)$ so that such an identity is sufficient in order to find an element r in R.

For $i \neq j$, we define $s(i,j)$ to be 0 if $i < j$ and 1 if $i > j$. We then have a sequence of *abelian groups*:

$$0 \to R \xrightarrow{(f_i)_{i=1}^k} \oplus_{i=1}^k R_{u_i} \xrightarrow{\left((-1)^{s(i,j)} g_{i,j}\right)_{i=1;j\neq i}^{k;k}} \oplus_{i=1;j>i}^{k-1;k} R_{u_i u_j}$$

It is a *complex* since the composite of successive homomorphisms is 0.

Patching Lemma: The above sequence is exact if the ideal $\langle u_1, \ldots, u_k \rangle$ is the unit ideal R in R.

This means that the first map identifies R with the kernel of the second map.

Proof of Patching Lemma

Since $\langle u_1, \ldots, u_k \rangle = R$ we have an identity $1 = \sum_{i=1}^k u_i x_i$ for some elements $x_1, \ldots, x_k$ in R. Taking the nk-th power of this, we obtain an identity $1 = \sum_{i=1}^k u_i^n t_i$ for some elements $t_1, \ldots, t_k$ in R. Hence, we have such an identity for every positive integer n.

First of all note that if $f_i(r) = 0$, then there is a positive integer n such that $u_i^n r = 0$. It follows that if $f_i(r) = 0$ for *all* $i = 1, \ldots, k$, then (since there are finitely many i) there is a common positive integer n such that $u_i^n r = 0$ for $i = 1, \ldots, k$. Multiplying both sides of the above identity by r, we see that $r = 0$. This shows that $R \to \oplus R_{u_i}$ is one-to-one.

Now suppose that we are given $(r_i)_{i=1}^k$ which satisfies $g_{i,j}(r_i) = g_{i,j}(r_j)$ for all i and j; in other words, this tuple is in the kernel.

Each r_i is of the form $s_i'/u_i^{n_i}$ for some element s_i' in R and non-negative integer n_i.

The given condition means that there is an $m_{i,j}$ such that the following identity holds *for elements of* R:

$$(u_i u_j)^{m_{i,j}} r_i = (u_i u_j)^{m_{i,j}} r_j$$

Since there are finitely many i and j, we can choose a fixed n so that:

- $r_i = s_i/u_i^n$ for an element s_i in R, and
- $(u_i u_j)^n r_i = (u_i u_j)^n r_j$ as elements of R.

Note that this means that $u_j^n s_i = u_i^n s_j$ as elements of R.

Multiplying the identity $1 = \sum_{j=1}^k u_j^n t_j$ by s_i, we obtain an identity in R:

$$s_i = \sum_{j=1}^k s_i u_j^n t_j = \sum_{j=1}^k u_i^n s_j t_j = u_i^n \left(\sum_{j=1}^k s_j t_j \right)$$

Now consider the element $r = \sum_{j=1}^k s_j t_j$ in R. By the above identity in R, we see that $0 = s_i - u_i^n r$ in R which means that $u_i^n r_i - u_i^n r = 0$ in R_{u_i}. By definition of equality in R_{u_i}, this means that $r = r_i$ in R_{u_i} as required. This completes the proof of exactness.

Patching homomorphisms

Given a commutative ring A and ring homomorphisms $h_i : A \to R_{u_i}$ such that $g_{i,j} \circ h_i = g_{j,i} \circ h_j$ we want to use the patching lemma to "lift" these uniquely a homomorphism $h : A \to R$.

We will use the following lemma from homological algebra:

Left-exactness of $\mathrm{Hom}(T, -)$**:** Given an abelian group T and an exact sequence of abelian groups

$$0 \to A \to B \to C$$

The resulting sequence of abelian groups of homomorphisms

$$0 \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, B) \to \mathrm{Hom}(T, C)$$

is also exact.

In particular, we see that given homomorphisms $h_i : A \to R_{u_i}$ satisfying $g_{i,j} \circ h_i = g_{j,i} \circ h_j$ as above, there is a unique *group* homomorphism $h : A \to R$ such that $h_i = f_i \circ h$. We need to check that this preserves multiplication.

We are given that h_i are ring homomorphisms for all i. Thus, given a and b in A we have $h_i(a)h_i(b) = h_i(ab)$. This means that $h(a)h(b)$ and $h(ab)$ have the same images under f_i for all i. As seen earlier, this means $h(a)h(b) = h(ab)$.

Application to Schemes

Given a $\mathbb{Z}$-affine scheme X and a commutative ring R, the collection $X(R)$ of R-points of X is identified with the set of ring homomorphisms $\mathrm{Hom}(\mathcal{O}(X), R)$. Moreover, a ring homomorphism $f : R \to S$ induces a set map $X(f) : X(R) \to X(S)$.

We can therefore interpret the above result by replacing A by $\mathcal{O}(X)$ as follows.

Sheaf property of Schemes: Given a commutative ring R and elements $u_1, \ldots, u_k$ which generate the unit ideal. Given R_{u_i}-points h_i of X for $i = 1, \ldots, k$ such that the image of h_i and h_j in $X(R_{u_i u_j})$ are the same, there is a unique point h in $X(R)$ so that h_i is its image in $X(R_{u_i})$