Barnali Jana

Will you please elaborate why there are only countably many $\Z$ affine Scheme ?

$(x_1, \dots, x_i, \dots)$

$A(x_1, \dots, x_p; \underbrace{f_1, \dots, f_q})$

$x_1^{m_1} \cdots x_p^{m_p}$

$f_i \in \underbrace{\mathbb{Z}[x_1, \dots, x_p]}_{\text{Countable set}} = \bigoplus_{\underline{m_1, \dots, m_p \in \mathbb{N}^p}} \mathbb{Z}(m_1, \dots, m_p)$

Finite subsets of a countable set

$\underline{\oplus k}$

$f_i \in k[x_1, \dots, x_p] = k$

$\mathcal{O}(X) \to \mathcal{O}(Y)$

$X(\mathcal{O}(Y))$

$\leftrightarrow Y \to X$

Ramanujan: Why do we want to view schemes as functors?

$\{X(R)\}_{R \text{ (any commutative ring)}}$

$\mathcal{O}(W) = \mathbb{Z}[x, y] / \langle x^2 + y^2 - 1 \rangle$

~~$R = \mathbb{Z}[$~~ $R = \mathbb{R}[t]/\langle t^2 \rangle = \underline{\mathbb{R}} \oplus \underline{\mathbb{R}}t$

$(a + bt)^2 + (c + dt)^2 - (1 + 0t) = 0$

$a, b, c, d \in \mathbb{R} \quad t^2 = 0$

$a^2 + 2abt + c^2 + 2cdt - 1 + 0t = 0$

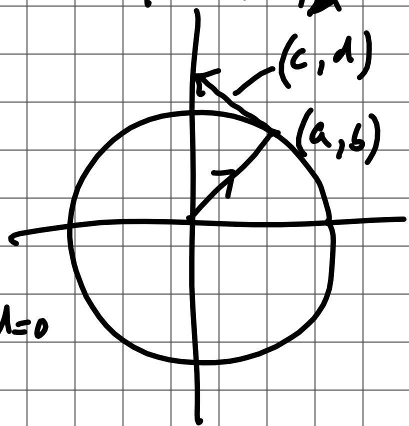
$a^2 + c^2 - 1 = 0$

$2ab + 2cd = 0 \quad ab + cd = 0$

$(a, c) \in S^1 : x^2 + y^2 = 1$

$W(\mathbb{R}[t]/t^2)$

$T W(\mathbb{R})$



Biplab Das

Difference Between affine variety and affine scheme ← Nilpotents & Zero divisors. play a non-trivial role in geometry. Grothendieck

Foundations

Importment ← | Chevalley . Zariski . Weil |

It seems like both are solution $\boxed{\text{set}}$? of some polynomial functions. So what's the difference in scheme theory?

Variety ↔ "Irreducible, reduced" scheme

$X_1 = A(x, y; xy)$ is a scheme & not a variety

$X_2 = A(x; x^2)$ " & "

$\mathcal{O}(X_1) = \frac{\mathbb{Z}[x,y]}{\langle xy \rangle}$

$\mathcal{O}(X_2) = \frac{\mathbb{Z}[x]}{\langle x^2 \rangle} \not\cong \frac{\mathbb{Z}[x]}{\langle x \rangle} = \mathbb{Z}$

$A(x; x, x+1)$ $\quad \mathbb{Z}[x]/\langle x, x+1 \rangle = \{0\}$

$\{0\}$ ring $\leftrightarrows$ empty scheme

ϕ is a topological space $\{\phi\} = \tau_\phi$

$\mathcal{O}(X)$ is a integral domain

$\boxed{1 \neq 0}$ & $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$

WHY FUNCTORS?

~~$X(R)$~~ $\quad R \to S$

$$X(R) \to X(S)$$

$R \to R$ automorphism.

$$\mathbb{Q}[i] = R \longrightarrow R = \mathbb{Q}[i]$$
$$i \longmapsto -i$$

$X(R) \circlearrowleft$

Galois actions on schemes!

$R = \mathbb{Z} \longrightarrow \mathbb{Z}/\langle p \rangle = \mathbb{F}_p$

$x^2 + y^2 = z^2 \longrightarrow$

$\downarrow$ Solutions over $\mathbb{Z}$

$\searrow$ Solutions over $\mathbb{F}_p$ for all p

Lifting question

If $a_p \in X(\mathbb{F}_p) \neq \emptyset$ for all p is there a $a \in X(\mathbb{Z})$ which lifts it.?

$\mathbb{Q}\mathbb{E}$

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r, "\times"] & X(\mathbb{F}_p) \\
X(\mathbb{Z}) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

essentially a bijection

Complete schemes.

FPR $\quad \dfrac{\mathbb{Z}[x_1, \ldots, x_p]}{\langle f_1, \ldots, f_p \rangle}$

Locally Ringed Topological spaces.

~~X~~ $\mathbb{Z}$-Aff. = FPR^{opp}

CRing $\xrightarrow{X_0}$ Set

} expand to include more functors as schemes

Category, Functor, Natural Transform

"Sets with structures"

Affine

$\downarrow$

Projective $\to$ More general schemes

Varieties

Functor captures the idea of "trans"

"Functor of points" — Google it!

Sundara

$X\left(k[t]/\langle t^n\rangle \right)$

higher order approximation

"Thick point"

$X = A\{x_1, x_2;\} = \mathcal{O}(X) = \mathbb{Z}[x_1, x_2]$

$k[x_1,x_2]/\langle x_1^2, x_1x_2, x_2^2\rangle$

different! $\langle x_1^2, x_2^2\rangle$

$$\left[\begin{array}{l} k[x_1,x_2]/\langle x_1^2, x_1x_2, x_2^2\rangle = k\oplus kx_1\oplus kx_2 \\ k[x_1,x_2]/\langle x_1^2, x_2^2\rangle = k\oplus kx_1\oplus kx_2\oplus kx_1x_2 \end{array}\right.$$

$k[x]/\langle f\rangle$ f is a poly of degree d

$\downarrow$

d dim'l v.s $/k$

$\downarrow$

"d points "counted correctly""

- $(x-a) \mid f$
- $(x-a)^2 \mid f$
- f is irred factor of degree e