Categories and Functors

We explore the notion of the category of $\mathbb{Z}$-affine schemes and functors on it.

$\mathbb{Z}$-affine schemes and morphisms

We introduced the notion of a $\mathbb{Z}$-affine scheme.

$\mathbb{Z}$-affine scheme: A $\mathbb{Z}$-*affine scheme* is of the form $A(x_1, \ldots, x_p; f_1, \cdots, f_q)$ where $f_1, \ldots, f_q$ are polynomials in the variables $x_1, \ldots, x_p$ with coefficients in the ring $\mathbb{Z}$ of integers.

To an affine scheme $X = A(x_1, \ldots, x_p; f_1, \cdots, f_q)$ we associated the commutative ring:

$$\mathcal{O}(X) = \frac{\mathbb{Z}[x_1, \ldots, x_p]}{\langle f_1, \ldots, f_q \rangle},$$

R-points of X: Give an affine scheme X and a commutative ring R, an R-point of X is a ring homomorphism $f : \mathcal{O}(X) \to R$.

A morphism of schemes can now be defined.

Morphism of $\mathbb{Z}$-affine schemes: Given $\mathbb{Z}$-affine schemes X and Y, a *morphism* $f : X \to Y$ is an $\mathcal{O}(X)$-point of Y.

We usually denote the corresponding ring homomorphism as $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ to indicate that it is in the *opposite* direction of the morphism of schemes.

We denote the collection of morphisms as $\mathrm{Mor}(X, Y)$ and so we have

$$\mathrm{Mor}(X, Y) = \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$$

where the latter is the collection of ring homomorphisms.

The important point to remember is the following.

> *All* properties of $\mathbb{Z}$-affine schemes are understood in terms of the above definitions.

More specifically, the notion of $\mathbb{Z}$-affine schemes and morphism between them determine a category.

Sets with structure

The definition we have given of a $\mathbb{Z}$-affine scheme is quite *different* from some of the definitions encountered elsewhere.

A typical (20-th century) mathematical definition is that of a "set with structure". For example:

Group: A group is a set G with an element 1_G and operations μ_G (multiplication) and ι_G (inverse) that satisfy some properties.

Topological space: A topological space is a set X with a collection τ_X of subsets (open subsets), that satisfy some properties.

. . . and so on.

We then define the associated "morphisms", or distinctive set maps as those that "preserve" the structure.

Group Homomorphism: Given groups G and H a group homomorphism is a set map $f : G \to H$ such that $f(1_G) = 1_H$, $f \circ \iota_G = \iota_H \circ f$ and $f \circ \mu_G = \mu_H \circ (f, f)$.

Continuous Map: Given topological spaces X and Y a continuous map is a set map $f : X \to Y$ such that $f^{-1}(U) \in \tau_X$ if $U \in \tau_Y$.

. . . and so on.

Categorical viewpoint

Category theory takes a different point of view:

> Mathematical structure is *determined* by morphisms; the "internal" set-theoretic structure of the objects is less (or not!) relevant.

In a category we have objects and morphisms. Let us denote objects by capital letters $X, Y, Z, \ldots$ and morphisms by lower-case letters $f, g, h, \ldots$.

- For every object X, we have an identity morphism $i_X : X \to X$.
- Given a morphism $f : X \to Y$ and a morphism $g : Y \to Z$, we can compose to get $g \circ f : X \to Z$.
- Given a morphism $f : X \to Y$ we have $i_Y \circ f = f = f \circ i_X$. In other words, the identity morphisms act as identity with respect to composition.
- Given morphisms $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$, we have the associativity of composition.$h \circ (g \circ f) = (h \circ g) \circ f$.

Note that we can make now make sense of some "standard" notions.

Isomorphism: A morphism $f : X \to Y$ is an isomorphism if there is a morphism $g : Y \to X$ such that $g \circ f = i_X$ and $f \circ g = i_Y$. This g is called an inverse of f in this case.

Exercise: Check that if $h : Y \to X$ is such that $h \circ f = i_X$, then $h = g$. This shows that the inverse is unique.

Exercise: Check that if $f : X \to Y$, $g : Y \to X$ and $h : Y \to X$ are such that $g \circ f = i_X$ and $f \circ h = i_Y$, then $g = h$ and all of these morphisms are isomorphisms.

Automorphism: An isomorphism $f : X \to X$ is called an *automorphism* of X.

Clearly, i_X is an automorphism. Moreover, the composition of automorphisms is an automorphism.

In many cases, the morphisms from X to Y form a set which is denoted by $\mathrm{Mor}(X, Y)$. In such cases we see that the subset $\mathrm{Aut}(X)$ of $\mathrm{Mor}(X, X)$ which consists of automorphisms, forms a group.

Standard Examples

- There is a category **Set** whose objects are sets and morphisms are set maps.
- There is a category **Gp** whose objects are groups and morphisms are group homomorphisms.
- There is a category **Top** whose objects are topological spaces and morphisms are continuous maps.
- There is a category **Ring** whose objects are rings with identity and morphisms are ring homomorphisms.

All these categories are "big" in the sense that objects are not members of a set. (Russell's paradox prevents us from talking about the set of all groups.) However, morphisms between two chosen objects *do* form a set in all these cases.

Other examples

- There is a category $\mathcal{F}$ whose objects are the sets $[n] = \{0, \ldots, n-1\}$ for a non-negative integer n (here $[0, 0-1]$ is interpreted as the empty set); a morphism $f : [n] \to [m]$ is just a map of (finite) sets.
- Given a field F, there is a category $\mathcal{V}_F$ whose objects are the sets F^n and a morphism $f : F^n \to F^m$ is an $m \times n$ matrix.

Note that in both these categories, the objects form a *countable* set. In the second case, if F is an uncountable field, then morphisms are also uncountable. Otherwise, the morphisms also form a countable set!

Secondly, note that, in some sense, the category $\mathcal{F}$ is *essentially* the category of *finite* sets. (However, Russell's paradox also prevents us from talking about the set of all finite sets!)

Similarly, the category $\mathcal{V}_F$ is *essentially* the category of finite dimensional vector spaces.

$\mathcal{FPR}$

Consider the category $\mathcal{FPR}$ whose objects are Finitely Presented commutative Rings:

$$\frac{\mathbb{Z}[x_1, \ldots, x_p]}{\langle f_1, \ldots, f_q \rangle}$$

where $x_1, \ldots, x_p$ are treated as "dummy" variables as seen earlier, and morphisms are ring homomorphisms.

By standard results in ring theory, we see that morphisms in $\mathcal{FPR}$ satisfy the properties expected for a category

Note that the category $\mathcal{FPR}$ has countably many objects (since we treat the variables as "dummy" using the semi-group approach to polynomials). Moreover, since a morphism of such rings is *determined* by the images of the variables, these are also countable.

$\mathbb{Z}$-Aff

We have the category $\mathbb{Z}$-Aff whose objects are $\mathbb{Z}$-affine schemes and morphisms are morphisms of $\mathbb{Z}$-affine schemes as defined above.

One can directly check the properties of morphisms as listed above. However, we will reduce this question to one we "know".

Opposite Category

Given a category $\mathcal{C}$, we consider the category $\mathcal{C}^{\text{opp}}$ whose objects are the same as the objects of $\mathcal{C}$ and morphisms are also the same as the morphisms of $\mathcal{C}$ *except* that we *reverse the arrows*!

To clarify, given an object X of $\mathcal{C}$, let X^{opp} denote the same object when considered in $\mathcal{C}^{\text{opp}}$. Given a morphism $f : X \to Y$ in $\mathcal{C}$, we denote by $f^{\text{opp}} : Y^{\text{opp}} \to X^{\text{opp}}$, the corresponding morphism in $\mathcal{C}^{\text{opp}}$. We define $i_{X^{\text{opp}}} = (i_X)^{\text{opp}}$.

One easily checks that morphisms in $\mathcal{C}^{\text{opp}}$ satisfy the properties expected for a category.

$\mathbb{Z}$-Aff is a category

We see that if X is a $\mathbf{Z}$-affine scheme, then $\mathcal{O}(X)$ is an object in $\mathcal{FPR}$. Conversely, given an object $\mathbb{Z}[x_1, \ldots, x_p]/\langle f_1, \ldots, f_q\rangle$ in $\mathcal{FPR}$, we have the associated $\mathbb{Z}$-affine scheme $A(x_1, \ldots, x_p; f_1, \ldots, f_q)$.

We have seen that a morphism $f : X \to Y$ of $\mathbf{Z}$-affine schemes corresponds *precisely* to a ring homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

It follows that $\mathbb{Z}$-Aff is $\mathcal{FPR}^{\text{opp}}$. In particular, we see immediately that ring theory has *already* proved that morphisms in $\mathbb{Z}$-Aff satisfy the properties expected for a category.

This may be seen as the basis of the statement:

> Affine algebraic geometry is the same as commutative algebra.

In this course, *schemes* will be the primary concept and thus we will look at everything through the "prism" of $\mathbb{Z}$-Aff.

Small Categories

When the objects of a category form a set C_0 and the morphisms C_1 also form a set, we say that the category is *small*.

In this case, we can see that a category can itself be written as "a set with structure".

A small category is:

- a set C_0 of objects
- a set C_1 of morphisms
- a map $i : C_0 \to C_1$ that takes an object X to its identity morphism
- maps $s, e : C_1 \to C_0$ that take a morphism to the "domain" and range" of the morphism.
- the subset C_2 of $C_1 \times C_2$ consisting of *all* pairs (g, f) such that $e(f) = s(g)$ (i.e. these are composable morphisms.
- a map $\circ : C_2 \to C_1$ that gives the composition of morphisms.

Exercise: Write down the properties of these maps that are required to define a (small) category.

Note that we actually *only* need the set C_1 as C_0 and C_2 can be determined from it. This leads to the "picture" of a small category as set of arrows which are "composable".

We can then the notion of a "map of categories that preserves the structure". Such a map is called a *functor* which we now define in greater generality.

Functors

Given categories $\mathcal{C}$ and $\mathcal{D}$, a functor F from $\mathcal{C}$ to $\mathcal{D}$:

- to an object X of $\mathcal{C}$ associates an object $F(X)$ of $\mathcal{D}$
- to a morphism $f : X \to Y$ of $\mathcal{C}$ associates a morphism $F(f) : F(X) \to F(Y)$ of $\mathcal{D}$.

such we have $F(i_X) = i_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Such a functor is sometimes called a *covariant* functor.

A functor from $\mathcal{C}^{\text{opp}}$ to $\mathcal{D}$ is called a *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$.

A contravariant functor F associates to a morphism $f : X \to Y$ of $\mathcal{C}$ to a morphism $F(f) : F(Y) \to F(X)$ of $\mathcal{D}$.

Functor of points

One important type of functor is the "functor of points". Given a $\mathbb{Z}$-affine scheme X we have seen that to each commutative ring R, we have associated a set $X(R)$ of R-points of X. We now claim that this is a functor. To avoid confusion, let us denote this functor as $X_.$ and define $X_.(R) = X(R)$.

Recall that there is a commutative ring $\mathcal{O}(X)$ associated with X so that there is a natural identification $X(R) = \mathrm{Hom}(\mathcal{O}(X), R)$.

Hence, an element $\mathbf{a} \in X(R)$ is identified with a honomorphism $\mathbf{a} : \mathcal{O}(X) \to R$.

Given a ring homomorphism $h : R \to S$ we obtain (by composition) a homomorphism $h \circ \mathbf{a} : \mathcal{O}(X) \to S$. This is an element of $X(S)$.

Hence, we see that $X_\cdot(h)$ given by $\mathbf{a} \mapsto h \circ \mathbf{a}$ is a set map $X_\cdot(R) \to X_\cdot(S)$.

Exercise: With definitions as above check that $X_\cdot$ is a functor from **CRing** to **Set**.

The basic idea is that *general* schemes will be *other* such functors. In other words, thinking of a $\mathbb{Z}$-affine scheme X in terms of the functor $X_\cdot$ will allow us to define more general schemes.

The functor $A^\cdot$

In fact, given a commutative ring A, we can define a functor $A^\cdot$ from **CRing** to **Set** as follows:

- For a ring we define $A^\cdot(R) = \mathrm{Hom}(A, R)$. Note that $\mathrm{Hom}(A, R)$ is a set!
- For a ring homomorphism $h : R \to S$, we define $A^\cdot(h) : A^\cdot(R) \to A^\cdot(S)$ by composition. Given $f : A \to R$ an element of $A^\cdot(R)$ we have $A^\cdot(h) = h \circ f : A \to S$ which is an element of $A^\cdot(S)$.

The associative property of composition and the right identity property of i_R show that this is a functor. We will see shortly how the left identity property of i_A gets used!

Note that $X_\cdot$ is the same as the functor $A^\cdot$ where $A = \mathcal{O}(X)$. This gives a proof that $X_\cdot$ is a functor.

Natural transformations

Given functors F and G from $\mathcal{C}$ to $\mathcal{D}$, we have the notion of a *natural transformation* $\eta : F \to G$.

This associates to each object X in $\mathcal{C}$ a morphism $\eta(X) : F(X) \to G(X)$ in $\mathcal{D}$ which has the property that if $f : X \to Y$ is a morphism in $\mathcal{C}$, then $\eta(Y) \circ F(f) = G(f) \circ \eta(X)$.

In other words, the following diagram commutes

$$\begin{array}{ccc} F(X) & \stackrel{\eta(X)}{\to} & G(X) \\ F(f)\downarrow & & \downarrow G(f) \\ F(Y) & \stackrel{\eta(Y)}{\to} & G(Y) \end{array}$$

Morphisms as natural transformations

Given X and Y are $\mathbb{Z}$-affine schemes, a morphism $f : X \to Y$ corresponds to a ring homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

For a ring R, given $\mathbf{a} : \mathcal{O}(X) \to R$, we can compose to get

$$\mathbf{a} \circ f^* : \mathcal{O}(Y) \to R$$

Thus, we have $\tilde{f}(R) : X(R) \to Y(R)$ for each ring R defined by $\tilde{f}(\mathbf{a}) = \mathbf{a} \circ f^*$ considered as an element of $Y(R)$.

Exercise: Check that $\tilde{f}$ is a natural transformation $X_\cdot \to Y_\cdot$ where these are considered as functors **CRing** to **Set**.

Yoneda Lemma for CRing

More generally, suppose F is a functor from **CRing** to **Set**.

One important (and elementary) result identifies, natural transformations $\eta : A^\cdot \to F$ with elements f of $F(A)$.

Given a natural transformation $\eta : A^\cdot \to F$, we note that $\eta(A) : A^\cdot(A) \to F(A)$ is a set map.

Applying this set map to $i_A \in A^\cdot(A)$ we have an element $f = \eta(A)(i_A) \in F(A)$ associated with η.

Conversely, given $f \in F(A)$, we define $\eta : A^\cdot \to F$ as follows. Given an object B in **CRing** and $g \in A^\cdot(B) = \mathrm{Hom}(A, B)$, the fact that F is a functor gives $F(g) : F(A) \to F(B)$. We then define $\eta(Y)(g) = F(g)(f)$.

Exercise: Check that η as defined above is a natural transformation.

In particular, we note that natural transformations $A^\cdot \to B^\cdot$ can be identified with $B^\cdot(A) = \mathrm{Hom}(B, A)$. We can use $f^\cdot : A^\cdot \to B^\cdot$ to denote the natural transformation associated with a ring homomorphism $f : B \to A$.

We can apply this to the functors $X_\cdot = A^\cdot$ where $A = \mathcal{O}(X)$ and $Y_\cdot = B^\cdot$ where $B = \mathcal{O}(Y)$. It follows that a natural transformation $X_\cdot \to Y_\cdot$ can be identified with a morphism $X \to Y$. (Note the *double* reversal!)

> The category $\mathbb{Z}$-Aff can be seen as a category of functors **CRing** to **Set** with morphisms between functors being defined as natural transformations.

Yoneda Lemma in general. One can observe that there is nothing special about **CRing** being used in the above result.

Given a category $\mathcal{C}$ for which, morphisms between a pair of objects X and Y form a set $\mathrm{Mor}(X, Y)$. For each object C of $\mathcal{C}$ we define a functor $C^\cdot$ from $\mathcal{C}$ to **Set** as follows:

- For an object X, we define $C^{\cdot}(X) = \mathrm{Mor}(C, Y)$.
- For a morphism $f : X \to Y$, we define $C^{\cdot}(f) : C^{\cdot}(X) \to C^{\cdot}(Y)$ by composition of morphisms.

Now consider *any* functor F from $\mathcal{C}$ to **Set**.

There is a natural identification between natural transformations $\eta : C^{\cdot} \to F$ and elements of $F(C)$ which sends η to $\eta(C)(i_C)$.

Conversely, given f in $F(C)$, we define $\eta : C^{\cdot} \to F$ as follows. Given g in $C^{\cdot}(X) = \mathrm{Mor}(C, X)$, we have $F(g) : F(C) \to F(X)$ since F is a functor. Hence, we have $F(g)(f)$ in $F(X)$. We use this to define $\eta(X)(g) = F(g)(f)$.

Conclusion

- We introduced the categories, functors and natural transformations.
- We provided some important examples of categories.
- In particular, we introduced the category $\mathbb{Z}$-Aff of $\mathbb{Z}$-Affine schemes.
- We also showed that a $\mathbb{Z}$-Affine scheme can be seen as a functor **CRing** to **Set**.
- The Yoneda lemma identifies morphisms between schemes as natural transformation of functors.
- This points the way to **extending** the category $\mathbb{Z}$-Aff to a bigger category of such functors.