Functors and Natural Transformations

MTH437 — Introduction to Schemes

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Recall

We introduced the notion of categories as a new viewpoint when compared with the notion of sets with structure.

The basic idea is that of a certain type of objects and morphisms between them which can be composed. We have identity morphisms and composition satisfies associativity.

We introduced examples of categories like **Set**, **Gp**, **Top**, **Ring**, **CRing** associated with these type of mathematical objects.

We introduce the *small* category $\mathbb{Z}$-Aff of $\mathbb{Z}$-Affine schemes which was the opposite category of $\mathcal{FPR}$ the category of finitely presented commutative rings.

Affine algebraic geometry is the dual notion of commutative algebra.

Today we will look at $\mathbb{Z}$-Aff slightly differently so that we can *extend* this category to define the category of schemes.

Functors

Given categories $\mathcal{C}$ and $\mathcal{D}$, a functor F from $\mathcal{C}$ to $\mathcal{D}$:

- to an object X of $\mathcal{C}$ associates an object $F(X)$ of $\mathcal{D}$,
- to a morphism $f : X \to Y$ of $\mathcal{C}$ associates a morphism $F(f) : F(X) \to F(Y)$ of $\mathcal{D}$,
- such that we have $F(i_X) = i_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Such a functor is sometimes called a *covariant* functor.

A functor from $\mathcal{C}^{\mathrm{opp}}$ to $\mathcal{D}$ is called a *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$.

For a *contravariant* functor, a morphism $f : X \to Y$ is associated to a morphism $F(f) : F(Y) \to F(X)$.

Functor of points

Given a $\mathbb{Z}$-affine scheme X we have seen that to each commutative ring R, we have associated a set $X(R)$ of R-points of X.

We now claim that this is gives a functor **CRing** to **Set**.

To avoid confusion, we will denote this functor as $X_{\cdot}$ and define $X_{\cdot}(R) = X(R)$.

Recall that there is a commutative ring $\mathcal{O}(X)$ associated with X so that there is a natural identification $X(R) = \mathrm{Hom}(\mathcal{O}(X), R)$.

Hence, an element $\mathbf{a} \in X(R)$ is identified with a honomorphism $\mathbf{a} : \mathcal{O}(X) \to R$.

Given a ring homomorphism $h : R \to S$ obtain (by composition) a homomorphism $h \circ \mathbf{a} : \mathcal{O}(X) \to S$. This is an element of $X(S)$.

Hence, we see that $X_{.}(h)$ given by $\mathbf{a} \mapsto h \circ \mathbf{a}$ is a set map $X_{.}(R) \to X_{.}(S)$.

Exercise: With definitions as above check that $X_{.}$ is a functor from **CRing** to **Set**.

We will give a more conceptual argument below.

The functor $A^{\cdot}$

In fact, given a commutative ring A, we can define a functor $A^{\cdot}$ from **CRing** to **Set** as follows:

- For a ring we define $A^{\cdot}(R) = \mathrm{Hom}(A, R)$. Note that $\mathrm{Hom}(A, R)$ is a set!
- For a ring homomorphism $h : R \to S$, we define $A^{\cdot}(h) : A^{\cdot}(R) \to A^{\cdot}(S)$ by composition. Given $f : A \to R$ an element of $A^{\cdot}(R)$ we have $A^{\cdot}(h) = h \circ f : A \to S$ which is an element of $A^{\cdot}(S)$.

The associative property of composition and the right identity property of i_R show that this is a functor. We will see shortly how the left identity property of i_A gets used!

Note that the functor $X_{\cdot}$ is the same as the functor $A^{\cdot}$ where $A = \mathcal{O}(X)$.

Natural transformations

Given functors F and G from $\mathcal{C}$ to $\mathcal{D}$, we have the notion of a *natural transformation* $\eta : F \to G$.

This associates, to each object X of $\mathcal{C}$, a morphism $\eta(X) : F(X) \to G(X)$ in $\mathcal{D}$.

This has the property that if $f : X \to Y$ is a morphism in $\mathcal{C}$, then $\eta(Y) \circ F(f) = G(f) \circ \eta(X)$.

In other words, the following diagram commutes

$$\begin{array}{ccc} & F(X) \stackrel{\eta(X)}{\to} G(X) & \\ F(f) \downarrow & & \downarrow G(f) \\ & F(Y) \stackrel{\eta(Y)}{\to} G(Y) & \end{array}$$

Morphisms as natural transformations

Given X and Y are $\mathbb{Z}$-affine schemes, a morphism $f : X \to Y$ corresponds to a ring homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

For a ring R, given $\mathbf{a} : \mathcal{O}(X) \to R$, we can compose to get

$$\mathbf{a} \circ f^* : \mathcal{O}(Y) \to R$$

Thus, we have $\tilde{f}(R) : X(R) \to Y(R)$ for each ring R defined by $\tilde{f}(R)(\mathbf{a}) = \mathbf{a} \circ f^*$ considered as an element of $Y(R)$.

Exercise: Check that $\tilde{f}$ is a natural transformation $X_{\cdot} \to Y_{\cdot}$ which are considered as functors **CRing** to **Set**.

Yoneda Lemma for **CRing**

More generally, suppose F is a functor from **CRing** to **Set**.

Given a natural transformation $\eta : A^{\cdot} \to F$, we note that $\eta(A) : A^{\cdot}(A) \to F(A)$ is a set map.

Applying to the element i_A in $A^{\cdot}(A)$, we have an element $f = \eta(A)(i_A) \in F(A)$ associated with η.

Conversely, given $f \in F(A)$, we define $\eta : A^{\cdot} \to F$ as follows.

Given an object B in **CRing** and $g \in A^{\cdot}(B) = \mathrm{Hom}(A, B)$, the fact that F is a functor gives $F(g) : F(A) \to F(B)$. We then define $\eta(Y)(g) = F(g)(f)$.

Exercise: Check that η as defined above is a natural transformation.

In particular, we note that natural transformations $A^{\cdot} \to B^{\cdot}$ can be identified with $B^{\cdot}(A) = \mathrm{Hom}(B, A)$.

We can use $f^{\cdot} : A^{\cdot} \to B^{\cdot}$ to denote the natural transformation associated with a homomorphism $f : B \to A$.

We can apply this to the functors $X_{\cdot} = A^{\cdot}$ where $A = \mathcal{O}(X)$ and $Y_{\cdot} = B^{\cdot}$ where $B = \mathcal{O}(Y)$.

It follows that a natural transformation $X_{\cdot} \to Y_{\cdot}$ can be identified with a morphism $X \to Y$. (Note the *double* reversal!)

The category $\mathbb{Z}$-Aff can be seen as a category of functors ***CRing*** *to* ***Set*** *with morphisms between functors being defined as natural transformations.*

The notes also explain that the Yoneda lemma is *not special* to **CRing**.

Conclusion

- We introduced the categories, functors and natural transformations.
- We provided some important examples of categories.
- In particular, we introduced the category $\mathbb{Z}$-Aff of $\mathbb{Z}$-Affine schemes.
- We also showed that a $\mathbb{Z}$-Affine scheme can be seen as a functor **CRing** to **Set**.
- The Yoneda lemma identifies morphisms between schemes as natural transformation of functors.
- This points the way to *extending* the category $\mathbb{Z}$-Aff to a bigger category of such functors.