Categories

MTH437 — Introduction to Schemes

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16th September 2021

Recall

We introduced the notion of a $\mathbb{Z}$-affine scheme.

$\mathbb{Z}$-**affine scheme**: A $\mathbb{Z}$-*affine scheme* is of the form $A(x_1, \ldots, x_p; f_1, \cdots, f_q)$ where $f_1, \ldots, f_q$ are polynomials in the variables $x_1, \ldots, x_p$ with coefficients in the ring $\mathbb{Z}$ of integers.

To an affine scheme $X = A(x_1, \ldots, x_p; f_1, \cdots, f_q)$ we associated the commutative ring:

$$\mathcal{O}(X) = \frac{\mathbb{Z}[x_1, \ldots, x_p]}{\langle f_1, \ldots, f_q \rangle},$$

R-point of X: Give an affine scheme X and a commutative ring R, an R-point of X is a ring homomorphism $f : \mathcal{O}(X) \to R$.

A morphism of schemes is now defined as follows.

Morphism of $\mathbb{Z}$-affine schemes: Given $\mathbb{Z}$-affine schemes X and Y, a *morphism* $f : X \to Y$ is an $\mathcal{O}(X)$-point of Y.

We usually denote the corresponding ring homomorphism as $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ to indicate that it is in the *opposite* direction of the morphism of schemes.

We denote the collection of morphisms as $\mathrm{Mor}(X, Y)$ and so we have

$$\mathrm{Mor}(X, Y) = \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$$

where the latter is the collection of ring homomorphisms.

The important point to remember is the following:

> *All properties of $\mathbb{Z}$-affine schemes are understood in terms of the above definitions.*

More specifically, the notion of $\mathbb{Z}$-affine schemes and morphisms between them determine a *category*.

Sets with structure

The definition we have given of a $\mathbb{Z}$-affine scheme is quite *different* from some of the definitions encountered elsewhere in mathematics courses.

A typical (20-th century) mathematical definition is that of a "set with structure". For example:

Group: A group is a set G with an element 1_G and operations μ_G (multiplication) and ι_G (inverse) that satisfy some properties.

Topological space: A topological space is a set X with a collection τ_X of subsets (open subsets), that satisfy some properties.

. . . and so on.

We then define the associated "morphisms", or distinctive set maps as those that "preserve" the structure.

Group Homomorphism: Given groups G and H a group homomorphism is a set map $f : G \to H$ such that $f(1_G) = 1_H$, $f \circ \iota_G = \iota_H \circ f$ and $f \circ \mu_G = \mu_H \circ (f, f)$.

Continuous Map: Given topological spaces X and Y a continuous map is a set map $f : X \to Y$ such that $f^{-1}(U) \in \tau_X$ if $U \in \tau_Y$.

. . . and so on.

Categorical viewpoint

Category theory takes a different point of view:

> *Mathematical structure is determined by morphisms; the "internal" set-theoretic structure of the objects is less (or not!) relevant.*

In a category we have objects and morphisms. Let us denote objects by capital letters $X, Y, Z, \ldots$ and morphisms by lower-case letters $f, g, h, \ldots$.

- For every object X, we have an identity morphism $i_X : X \to X$.
- Given a morphism $f : X \to Y$ and a morphism $g : Y \to Z$, we can compose to get $g \circ f : X \to Z$.
- Given a morphism $f : X \to Y$ we have $i_Y \circ f = f = f \circ i_X$. In other words, the identity morphisms act as identity with respect to composition.
- Given morphisms $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$, we have the associativity of composition.$h \circ (g \circ f) = (h \circ g) \circ f$.

Isomorphisms and Automorphisms

We can make now make sense of some "standard" notions.

Isomorphism: A morphism $f : X \to Y$ is an isomorphism if there is a morphism $g : Y \to X$ such that $g \circ f = i_X$ and $f \circ g = i_Y$. This g is called an inverse of f in this case.

Exercise: Check that if $h : Y \to X$ is such that $h \circ f = i_X$, then $h = g$. This shows that the inverse is unique.

Exercise: Check that if $f : X \to Y$, $g : Y \to X$ and $h : Y \to X$ are such that $g \circ f = i_X$ and $f \circ h = i_Y$, then $g = h$ and all of these morphisms are isomorphisms.

Automorphism: An isomorphism $f : X \to X$ is called an *automorphism* of X.

Clearly, i_X is an automorphism. Moreover, the composition of automorphisms is an automorphism.

In many cases, the morphisms from X to Y form a set which is denoted by $\mathrm{Mor}(X, Y)$. In such cases we see that the subset $\mathrm{Aut}(X)$ of $\mathrm{Mor}(X, X)$ which consists of automorphisms, forms a group.

Standard Examples

- There is a category **Set** whose objects are sets and morphisms are set maps.
- There is a category **Gp** whose objects are groups and morphisms are group homomorphisms.
- There is a category **Top** whose objects are topological spaces and morphisms are continuous maps.
- There is a category **Ring** whose objects are rings with identity and morphisms are ring homomorphisms.
- There is a category **CRing** whose objects are Commutative rings with identity and morphisms are ring homomorphisms.

All these categories are “big” in the sense that objects are not members of a set.

Russell’s paradox prevents us from talking about the set of all sets.

However, morphisms between two chosen objects *do* form a set in all these cases.

Other examples

- There is a category $\mathcal{F}$ whose objects are the sets $[n] = \{0, \ldots, n-1\}$ for a non-negative integer n (here $[0]$ is interpreted as the empty set); a morphism $f : [n] \to [m]$ is just a map of (finite) sets.
- Given a field F, there is a category $\mathcal{V}_F$ whose objects are the sets F^n and a morphism $f : F^n \to F^m$ is an $m \times n$ matrix.

Note that in both these categories, the objects form a *countable* set. In the second case, if F is an uncountable field, then morphisms are also uncountable. Otherwise, the morphisms also form a countable set!

Secondly, note that, in some sense, the category $\mathcal{F}$ is *essentially* the category of *finite* sets. (However, Russell's paradox also prevents us from talking about the set of all finite sets!)

Similarly, the category $\mathcal{V}_F$ is *essentially* the category of finite dimensional vector spaces.

$\mathcal{FPR}$

Consider the category $\mathcal{FPR}$ whose objects are Finitely Presented commutative Rings:

$$\frac{\mathbb{Z}[x_1, \dots, x_p]}{\langle f_1, \dots, f_q \rangle}$$

where $x_1, \dots, x_p$ are treated as "dummy" variables as seen earlier, and morphisms are ring homomorphisms.

By standard results in ring theory, we see that morphisms in $\mathcal{FPR}$ satisfy the properties expected for a category.

Note that the category $\mathcal{FPR}$ has countably many objects (since we treat the variables as "dummy" using the semi-group approach to polynomials). Moreover, since a morphism of such rings is *determined* by the images of the variables, these are also countable.

$\mathbb{Z}$-Aff

We have the category $\mathbb{Z}$-Aff whose objects are $\mathbb{Z}$-affine schemes and morphisms are morphisms of $\mathbb{Z}$-affine schemes as defined above.

One can directly check the properties of morphisms as listed above. However, we will reduce this question to one we “know”.

Opposite Category

Given a category $\mathcal{C}$, we consider the category $\mathcal{C}^{\rm opp}$ whose objects are the same as the objects of $\mathcal{C}$ and morphisms are also the same as the morphisms of $\mathcal{C}$ *except* that we *reverse the arrows*!

- Given an object X of $\mathcal{C}$, let $X^{\rm opp}$ denote the same object when considered in $\mathcal{C}^{\rm opp}$.
- Given a morphism $f : X \to Y$ in $\mathcal{C}$, we denote by $f^{\rm opp} : Y^{\rm opp} \to X^{\rm opp}$, the corresponding morphism in $\mathcal{C}^{\rm opp}$.
- We define $i_{X^{\rm opp}} = (i_X)^{\rm opp}$.

One easily checks that morphisms in $\mathcal{C}^{\rm opp}$ satisfy the properties expected for a category.

$\mathbb{Z}$-Aff is a category

We see that if X is a $\mathbb{Z}$-affine scheme, then $\mathcal{O}(X)$ is an object in $\mathcal{FPR}$.

Conversely, given an object $\mathbb{Z}[x_1, \ldots, x_p]/\langle f_1, \ldots, f_q \rangle$ in $\mathcal{FPR}$, we have the associated $\mathbb{Z}$-affine scheme $A(x_1, \ldots, x_p; f_1, \ldots, f_q)$.

We have seen that a morphism $f : X \to Y$ of $\mathbb{Z}$-affine schemes corresponds *precisely* to a ring homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

It follows that $\mathbb{Z}$-Aff is $\mathcal{FPR}^{\mathrm{opp}}$.

In particular, we see immediately that ring theory has *already* proved that morphisms in $\mathbb{Z}$-Aff satisfy the properties expected for a category.

This may be seen as the basis of the statement:

Affine algebraic geometry is the dual notion of commutative algebra.

In this course, *schemes* will be the primary concept and thus we will look at everything through the "prism" of $\mathbb{Z}$-Aff.