Examples of equation coefficients

We look at some examples to see how coefficients of equations can be "turned into variables".

$x^2 + \sqrt{2}y^2 = 1$

We introduce a new variable z and use the system of equations

$$\begin{aligned} x^2 + zy^2 - 1 &= 0 \\ z^2 &= 2 \end{aligned}$$

Note that this *also* has the solutions where $z = -\sqrt{2}$.

However, *algebraically* there is no distinction between these two solutions!

$x^3 + \pi y^2 = 1$

We *know* that π is transcendental. So we can treat it as a new variable z and there is only one equation

$$x^3 + zy^2 - 1 = 0$$

Note that this *also* has the solutions where $z = e$ (or some other transcendental number).

Again, *algebraically* there is no distinction between e and π!

Note that there are many solutions to this system of equations. All we can say is that the solutions to the original equation are *contained* in the solutions here for a *specific* value of z.

$\pi x^2 + \sqrt{\pi + 2e}y^2 + ez^2 = 1$

We introduce the variable u (respectively v) in place of π (respectively e). Note that we do not *know* that π and e are algebraically independent, so we just assume it! We also introduce w which satisfies the equation $w^2 = \pi + 2e$. We then have the system of equations

$$\begin{aligned} ux^2 + wy^2 + vz^2 - 1 &= 0 \\ w^2 - u - 2v &= 0 \end{aligned}$$

Note that there are many solutions to this system of equations.All we can say is that the solutions to the original equation are *contained* in the solutions here for *specific* values of u, v, w.

$\frac{\pi}{e^2+1}x^2 + \pi e y^2 = 1$

We introduce the variable u (respectively v) in place of π (respectively e). We also introduce the variable w which satisfies $w(e^2+1) = 1$. We have the system of equations

$$\begin{aligned} wux^2 + uvy^2 - 1 &= 0 \\ w(v^2+1) - 1 &= 0 \end{aligned}$$

$x^2/2 + y^2/3 = 1$

We introduce variables u and v that satisfy $2u = 1$ and $3v = 1$. The system of equations

$$\begin{aligned} ux^2 + vy^2 - 1 &= 0 \\ 2u - 1 &= 0 \\ 3v - 1 &= 0 \end{aligned}$$

contains all solutions to our original system of equations.

$x^2 + y^2 = 1$ over $\mathbb{F}_5$.

We *add* the equation $5 = 0$! The system of equations

$$\begin{aligned} x^2 + y^2 - 1 &= 0 \\ 5 &= 0 \end{aligned}$$

contains all solutions to our original system of equations.