The Impossible Triangle

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In the Mahabharata, when Duryodhana visits the palace at Indraprastha, he is confounded by floor tiles that perfectly resemble lotus ponds. I wonder how Duryodhana would feel if he wandered into the world of geometry - where mathematicians calmly discuss creatures as weirdly fantastic as the Impossible Triangle!

We all know that a triangle is a two-dimensional geometrical figure, with three vertices and with its three angles summing to 180^0. I can draw such a triangle with pen and ink on paper. I can also take three beams of wood, nail them together at the same angles, and construct a solid three-dimensional "real" object corresponding to this triangle. If I now make a drawing of this three-dimensional structure on paper, it will look like a solid object. This happens because our brain is able to reconstruct a solid shape from its two-dimensional representation on paper, by using our visual system's perception of depth. This it does by assigning a certain depth to each point in the picture and by conforming to certain rules for this representation. For example, if one looks at the solid shape of an object from its drawing on paper, certain lines appear on the plane of the paper while others appear to project out of the plane of the paper, in order to suggest depth. This is how the visual system is trained to translate two-dimensional pictures into three-dimensional images in the brain.

So what can be so impossible about a triangle anyway? To see this, now look at the triangle in Figure 1. What is wrong with it? If you look at each corner, it seems a perfectly good construction. What happens if you look at the triangle as a whole? Start looking from one corner and proceed along your line of vision to the next corner - the beams of the triangle seem to be approaching you and receding away from you, at the same time! Seen separately, the corners of the triangle seem alright, but do not make sense when seen in relation to each other.

You can see how your brain produces a spatial paradox when you separate and try to rejoin the triangle as in the steps given below.

1. Cover any corner of the triangle with your hand so that you see only two-thirds of the impossible triangle (as shown in Figure 2).

2. You will see a broken model of the triangle but notice that your perception of the shape of the triangle has now changed.

3. Similarly cover the other two corners of the triangle and see how the way its shape in your brain changes.

So the paradox disappears when you break or close off any part of the parallel lines that form the boundary of one beam. The straight lines are no longer seen as part of a continous surface boundary. If you break the impossible triangle by covering it, you will find that you are able to recreate the true picture in your brain: two arms are joined together at a corner, but the third arm does not join them; instead it extends out of the plane. So now you "know" that the Impossible Triangle your brain insists on seeing does not "really" exist. Take your hand away however, and the illusion comes right back. It is impossible for your brain to get rid of its surface representation of an image, although you know it is literally impossible and you have tried to "prove" it to your brain by giving it the correct spatial perspective and teaching it about the object!

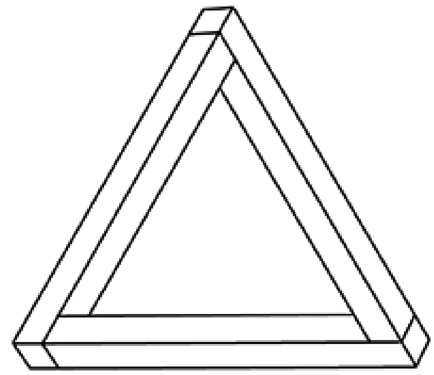

FIG. 1: The Impossible Triangle.

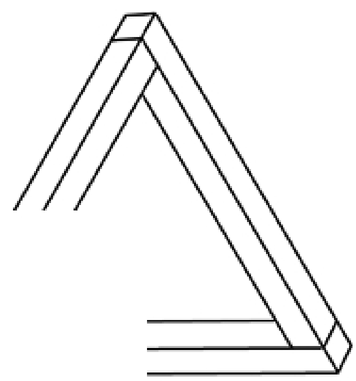

FIG. 2: Breaking the Impossible Triangle.

As we have seen, the Impossible Triangle is only a two-dimensional drawing and cannot exist in real life. It appears to be a solid object because your brain will always

FIG. 3: A real life model of the Impossible Triangle.

try to recreate a two-dimensional object in three dimensions, however paradoxical and confusing the end result may be! It is possible to construct a solid object that looks like an Impossible Triangle, but this will necessarily have to have a break in it or have one arm as a curved shape like you see in Figure 3; however, when viewed from a particular angle, it will look like a "real" triangle.

M. C. Escher (1898-1972) was an artist who made vivid lithographs filled with intriguing geometric constructs, vividly imagined creatures and puzzling mathematical patterns[1]. In the 1950's Roger Penrose (a British mathematician) was looking for something to use as an example of an impossibility in its purest and simplest form. The Impossible Triangle had already been used by European artists in their works; Penrose rediscovered it and called it the Penrose Tribar. While Escher was an artist who had no formal training in mathematics, lots of mathematicians delight in the way he used intricate and often puzzling mathematical concepts to create works of art. Let us see how Escher used the Impossible Triangle and other impossible geometrical constructions to create illusions in two of his works: *The Waterfall* and *Belvedere*. We know that any representation of a three-dimensional real object is the projection of it onto a flat surface (like a drawing on a sheet of paper). Escher however wanted to show us that the reverse is not always true - so every representation does not have to be a projection of a three-dimensional reality!

Try following the path of the water in Escher's famous lithograph *The Waterfall*, shown in Figure 4. We begin with a reality which seems perfectly reasonable - we follow the water as it flows down a ramp between two towers through levels marked by descending stone steps. By the time we reach the end we have reached a logical impossibility - the water seems to have flowed uphill since we have reached the top of the building and the water falls down a waterfall and drives a mill! The two towers are equally high, but the right one seems one level below the left one. If you try making an actual model of Escher's Waterfall, you will see that the vertical rods are actually curved S-shaped rods and the entire illusion is constructed by joining three Impossible Triangles along the water stream.

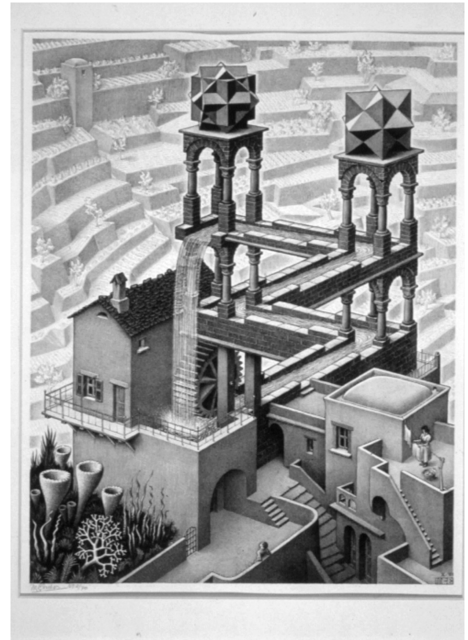

FIG. 4: Escher's famous lithograph *The Waterfall*[2].

Figure 5 shows another of Escher's works: this one, called *Belvedere*, shows a building with an impossible architecture. The same trick has been used to deceive our brains! Look at the weird eight pillars that join the two storeys of the building. Only the pillars on the extreme right and the extreme left behave "normally", the other six keep on joining front side to rear side and seem to somehow or the other pass diagonally through the space in the middle. This is shown by the ladder's top end leaning against the outer edge of the building, while its foot is inside. So where do you think the man halfway up the ladder is? Is he inside the building (yes, if you look from below) or outside it (yes, if you look from above)?! If we cut the picture horizontally through the centre, both halves are normal - only when we join it as Escher does, do we realise it is impossible.

Here are some exercises for you to have fun with!

1. Take three small wooden beams (or you can use thick cardboard if you wish). Nail together (or paste) the beams so that they form an actual impossible triangle, with one arm extending in space onto a different plane, and never meeting the other two. Look closely at your model: rotate it, move away from it, and view it from different angles till you find a perspective from where it looks like a triangle to you. Take a photo of it from this angle and see if your perspective matches your camera's "eye".

2. Try constructing an Impossible Cube, based on the same principle as the Impossible Triangle. **Hint:-** Use a magnifying glass and look carefully at the object the

man sitting to the left in the Figure *Belvedere* is holding in his hand!

3. Look at Figure 6 which shows another famous lithograph by Escher called *Ascending and Descending*. Here is a “continuous quasi-endless” staircase that goes upward or downward without getting higher or lower. If you follow any man in the figure, each pace takes him one step higher; yet on completing the circuit, he is back where he started! Can you figure out what impossible geometrical construction Escher has used to cheat your mind in this work of art?

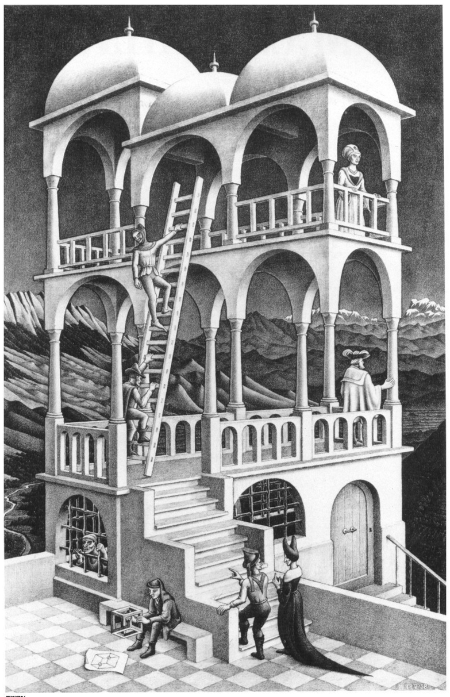

FIG. 5: Escher’s lithograph *Belvedere*[2].

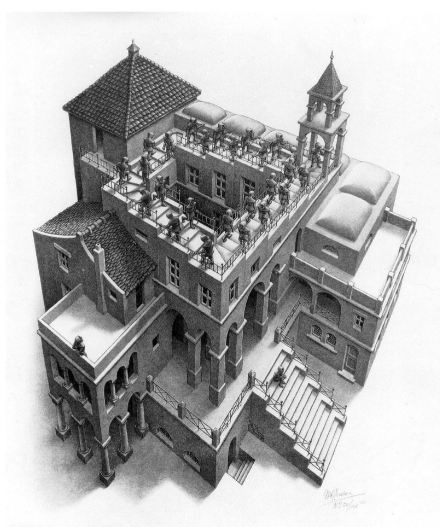

FIG. 6: Another of Escher’s famous lithographs *Ascending and Descending*[2].

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[1] A flat surface of stone or metal is treated to absorb or repel ink in a desired pattern. The pattern is then transferred to paper and printed as a lithograph.

[2] Reproduced with permission from `http://www.escher.freeserve.co.uk/escher`

[3] For more fun with geometry and puzzles visit the *Geometry Junkyard* website at `http://www.ics.uci.edu/ eppstein/junkyard/`.

[4] There are of course innumerable books on M. C. Escher and his works. You can use the books given below as a nice introduction to them. The magic mirror of M. C. Escher, Bruno Ernst, Taschen America Inc., New York (1994). The magic of M. C. Escher, with an introduction by J. L. Locher; designed by Erik The, Harry Abrams, New York (2000).