# DYNAMICAL SYSTEMS FOR LATTICES

KAPIL HARI PARANJAPE

A short write-up on the modular dynamical system based on lectures by Etienne Ghys.

## 1. Lattices and Matrices

A lattice $L$ in the plane is the collection of all *integer* linear combinations of a pair of linearly independent vectors, i. e.

$$L = \left\{ m \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + n \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} : m, n \text{ integers} \right\}$$

We can thus think of the lattice $L$ as the image of multiplication by a matrix as follows:

$$\begin{pmatrix} m \\ n \end{pmatrix} \mapsto m \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + n \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}$$

The condition of linear independence is $x_1y_2 - x_2y_1 \neq 0$, i. e. the determinant of the matrix is non-zero.

The same lattice can be given by a different matrix

$$\begin{pmatrix} kx_1 + lx_2 & mx_1 + nx_2 \\ ky_1 + ly_2 & my_1 + ny_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} k & l \\ m & n \end{pmatrix}$$

when $k$, $l$, $m$, $n$ are integers with $kn - lm = \pm 1$.

This can be understood in terms of group theory as follows. The collection $\mathrm{GL}(2, \mathbf{R})$ of $2 \times 2$ matrices with non-zero determinant forms a group. The subset $\mathrm{GL}(2, \mathbf{Z})$ of $\mathrm{GL}(2, \mathbf{R})$ which consists of matrices with integer entries *and* determinant $\pm 1$ is a subgroup. We have thus identified the collection of lattices with the *coset space* $\mathrm{GL}(2, \mathbf{R})/\mathrm{GL}(2, \mathbf{Z})$.

The determinant of a matrix is related to the area of the fundamental parallelogram of the lattice $L$.

**Exercise 1.** Show that the area of the parallelogram

$$P = \{(ax_1 + bx_2, ay_1 + by_2) : 0 \leq a, b \leq 1\}$$

is given by $|x_1y_2 - x_2y_1|$.

The subset $\mathrm{SL}(2, \mathbf{R})$ of $\mathrm{GL}(2, \mathbf{R})$ consisting of matrices of determinant 1 is also a subgroup. Moreover, the intersection $\mathrm{SL}(2, \mathbf{R}) \cap \mathrm{GL}(2, \mathbf{Z})$ of the two subgroups of $\mathrm{GL}(2, \mathbf{R})$ is the subgroup $\mathrm{SL}(2, \mathbf{Z})$ consisting of $2 \times 2$ integer matrices with determinant 1.

Hence we see that the coset space $\mathrm{SL}(2, \mathbf{R})/\mathrm{SL}(2, \mathbf{Z})$ can be identified with the collection of lattices whose fundamental parallelogram has area 1. Let us call this the space of *lattices of co-area 1.* In the sections that follow we will give a different decription of these spaces.

## 2. Lattice Invariants

In this section will identify the plane with the Gaussian complex plane.

A lattice $L$ is then the collection of all *integer* linear combinations of a pair of complex numbers $\omega_1$, $\omega_2$; here $\omega_2$ is not in the line $\mathbf{R}\,\omega_1$ which passes through $\omega_1$. We can re-write this condition as $(\omega_2/\omega_1) \notin \mathbf{R}$; in other words, the imaginary part $\mathrm{Im}(\omega_2/\omega_1)$ is non-zero.

**Exercise 2.** We can express $\omega_i$ in terms of their real and imaginary parts as follows

$$\omega_1 = x_1 + y_1\sqrt{-1} \text{ and } \omega_2 = x_2 + y_2\sqrt{-1}$$

Show that the imaginary part of $(\omega_2/\omega_1)$ is

$$\mathrm{Im}(\omega_2/\omega_1) = \frac{x_1y_2 - x_2y_1}{x_1^2 + y_1^2}$$

In particular, the condition of linear independence above is the same is the same as the usual condition $x_1y_2 - x_2y_1 \neq 0$, i. e. the determinant of the matrix is non-zero.

Through his study of "elliptic functions", Eisenstein was led to define the series

$$E_k(L) = \sum_{\omega\in L\setminus\{0\}} \frac{1}{\omega^k}$$

associated with the lattice $L$.

**Exercise 3.** Note that the series converges for $k \geq 4$. Moreover, $E_k(L) = 0$ for $k$ odd.

Continuing this work, Weierstrass was able to show that:

(1) A certain "discriminant"

$$\Delta(L) = 49E_6(L)^2 - 20E_4(L)^3$$

is *non-zero* for every lattice.

(2) Conversely, given *any* pair of complex numbers $(a, b)$ so that $\Delta(a, b) = 49b^2 - 20a^3 \neq 0$, there is a *unique* lattice $L$ such that $(a, b) = (E_4(L), E_6(L))$.

We will not prove this result as it would take us far afield into the theory of elliptic functions.

It follows that the "space of all lattices" can be identified with the collection of pairs $(a, b)$ of complex numbers that satisfy $\Delta(a, b) \neq 0$ via the map

$$L \mapsto (E_4(L), E_6(L))$$

Recall that we identifed the space of lattices with the coset space $\mathrm{GL}(2,\mathbf{R})/\mathrm{GL}(2,\mathbf{Z})$. We have thus obtained an explicit represention of this space as

$$\left(\mathbf{C}^2 \setminus \{\Delta = 0\}\right) = \{(a,b) : \Delta(a,b) \neq 0\}$$

We now want to apply this to lattices with co-area 1.

**Exercise 4.** Given a lattice $L$, show that there is a *unique* real number $k$ (which depends on $L$) so that $(E_4(kL), E_6(kL))$ lies on "the three dimensional sphere"

$$S^3 = \{(a,b) : |a|^2 + |b|^2 = 1\}$$

(**Hint**: When we replace a lattice $L$ by its multiple $kL$

$$(E_4(kL), E_6(kL)) = \left(k^{-4}E_4(L), k^{-6}E_6(L)\right)$$

This can be shown directly using the formulas.)

For a lattice $L$ of co-area 1, let $k(L)$ denote this number and consider the map

$$L \mapsto \left(E_4\left(k(L)L\right), E_6\left(k(L)L\right), \right)$$

This gives a map $\mathrm{SL}(2,\mathbf{R})/\mathrm{SL}(2,\mathbf{Z}) \to S^3$. If two lattices $L$ and $L'$ have the same image, then $k(L)L = k(L')L'$, so that $L' = \frac{k(L)}{k(L')}L$ is the multiple of $L$ by a real number. If $L$ and $L'$ both have co-area 1, we see that this real number must be $\pm 1$. In other words, $L' = \pm L = L$. As a consequence, we have produced a representation of the space of lattices of co-area 1 as a subset of $S^3$:

$$\begin{aligned}\mathrm{SL}(2,\mathbf{R})/\mathrm{SL}(2,\mathbf{Z}) &\cong \left(S^3 \setminus \{\Delta = 0\}\right) \\ &= \{(a,b) : |a|^2 + |b|^2 = 1 \text{ and } \Delta(a,b) \neq 0\}\end{aligned}$$

**Exercise 5.** In order to "see" $S^3$ more clearly exclude one point (say $(1,0)$) on it and use "stereographic projection" from this point to identify the complement with usual 3 dimensional space $\mathbf{R}^3$.

Under this projection, one can show that the set $\{\Delta = 0\} \cap S^3$ maps to a "Trefoil knot", which is the simplest knot one can make.

## 3. A dynamical system

Given a lattice $L$ of co-area 1, there is a natural way to shrink it in the $y$-direction and stretch it in the $x$-direction so that the resulting lattice is again of co-area 1. In terms of matrices this amounts to

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 & \alpha x_2 \\ \alpha^{-1} y_1 & \alpha^{-1} y_2 \end{pmatrix}$$

Since $L = -L$, we may restrict our attention to *positive* $\alpha$. In that case we can find a real number $t$ so that $\alpha = e^t$. Thinking of $t$ as a "time parameter", we can think of this transformation as giving a "flow" $L \mapsto \phi_t(L)$ in the space of lattices of co-area 1.

We can now use the identification given in the previous section to make this a flow on $S^3$ as follows. Given $(a, b)$ in $S^3$, let $L(a, b)$ denote the associated lattice of co-area 1 so that

$$(a, b) = (E_4\,(k \cdot L(a, b))\,, E_6\,(k \cdot L(a, b)))$$

where $k = k(L(a, b))$. (We will ignore the fact that for $\Delta(a, b) = 0$, there is no such $L$; this problem can be resolved with some jugglery!) At time $t$ we send $(a, b)$ to the image of $\phi_t(L(a, b))$ under the identification

$$L \mapsto (E_4\,(k(L)L)\,, E_6\,(k(L)L))$$

The resulting flow on $S^3$ is a called the "modular" dynamical system on $S^3$.

**Exercise 6.** Note that this is a *different* flow from that given by "right multiplication"

$$(\omega_1, \omega_2) \mapsto (e^t\omega_1, e^{-t}\omega_2)$$

The closed orbits of the modular flow have some interesting connections with number theory and the theory of binary quadratic forms as investigated by Gauss!