Varieties defined over Number Fields

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The Institute of Mathematical Sciences
Chennai

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Hilbert on Algebraic Geometry

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Since I like geometry, this talk will focus on finding graphical representations of algebraic equations!

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The variety defined is the locus of *simultaneous* solutions of this system in projective space $\mathbb{P}^n$.

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The varieties defined over number fields have special geometric properties

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- Given a zero-cycle of the form $\sum_i (P_i - Q_i)$ and a 1-form ω on the variety, the expression $\sum_i \int_{P_i}^{Q_i} \omega$ is well-defined upto periods. We say that a zero-cycle is Abel-Jacobi trivial if this is zero for all ω.
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- One can show that if the field has *just one* transcendental element, then there are zero-cycles such that *no multiple* is rationally trivial.

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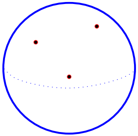

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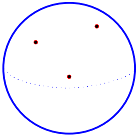

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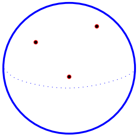

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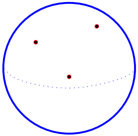

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This gives us a (as yet mysterious) relation between subgroups of finite index in this group and certain curves defined over number fields.

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On the other hand ...
The cross-ratio of four points

$$\lambda = \frac{z_\lambda - z_0}{z_1 - z_0} \times \frac{z_1 - z_\infty}{z_\lambda - z_\infty}$$

is unchanged under Möbius transformations.

Re-statement of Belyi's Theorem

Theorem (Belyi's Theorem)

A curve is defined over a number field

if and only if

it is a covering of the projective line which is unramified outside a rigid configuration of points.

Equi-singular deformations

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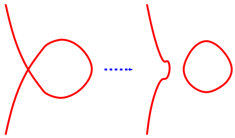

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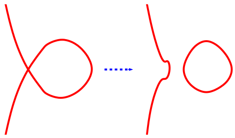

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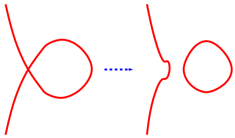

We want to restrict to deformations that do not change the topology. One way to achieve this is to restrict to equi-singular deformations.

Geometrically rigid figures

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if any equi-singular deformation of it is equivalent to it under an automorphism of $\mathbb{P}^n$. **Question**: What are all geometrically rigid curves

(configurations) in the projective plane?

Characterisation of Geometric Rigidity

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Theorem (Main Corollary)

A projective surface is defined over a number field

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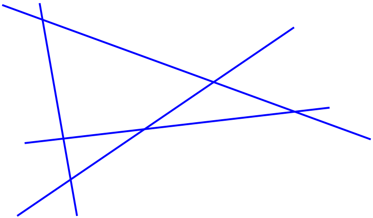
it is the covering of the plane which is unramified outside a geometrically rigid configuration.

Drawings of equations-1

The configuration of four lines in the plane is geometrically rigid.



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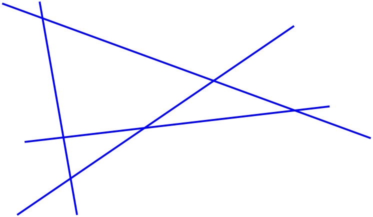
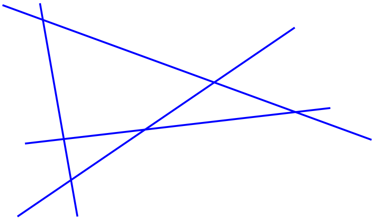


We can take take the equation of this configuration to be $XYZ(X+Y-Z)=0$.

Drawings of equations-2

Adding a line which passes through the points of intersection gives another rigid configuration.



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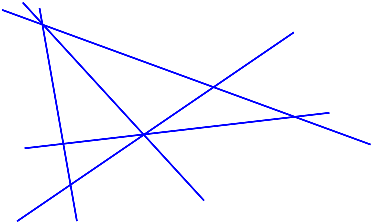

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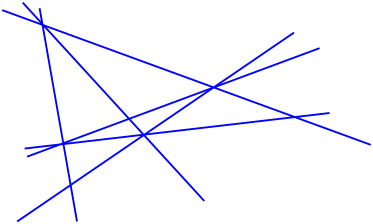

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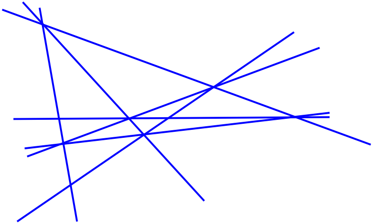

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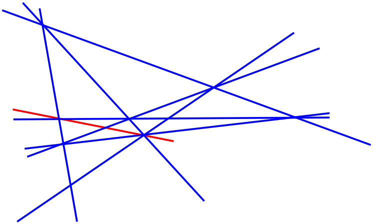

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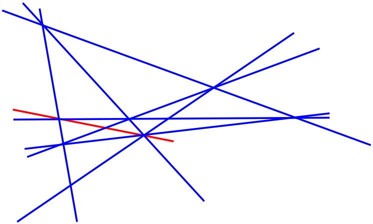

This gives a configuration which contains the line $2X + Y = Z$ (in red). We can similarly add *every* line defined over rationals.

Idea of the proof

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- A curve defined over a number field is uniquely determined by the points on it with co-ordinates in the field of algebraic numbers.
- Hence we can obtain a rigid configuration which contains such a curve.