THE WORK OF EFIM ZELMANOV (FIELDS MEDAL 1994)

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1. Introduction

Last year at the International Congress of Mathematicians (ICM '94) in Zürich, Switzerland one of the four recipients of the Fields Medal was a mathematician from the former Soviet Union—Prof. Efim Zelmanov, now at the University of Wisconsin, USA. He has used techniques from the theory of non-commutative rings to settle a problem in group theory known as the Restricted Burnside Problem. In the following article I attempt to give a flavour of this Problem and the method of its final resolution. The material is largely based on the talk given by Zelmanov given at the ICM '90 held in Kyoto, Japan[13].

2. Groups

For basic definitions and results of group theory please see a standard text such as [2] or [8].

In a finite group G every element g satisfies $g^n = e$ for some least positive integer n called the order of g and denoted by $\circ(g)$. This leads us to,

Problem 1 (General Burnside Problem)**.** Let G be a finitely generated group such that for every element g of G there is a positive integer N_g so that $g^{N_g} = e$. Then is G finite?

When G arises as a group of $n \times n$ matrices (or more formally when G is a linear group) it was shown by Burnside that the answer is yes (a simple proof is outlined in Appendix A). However, in 1964 Golod and Shafarevich [3] showed that this is not true for all groups. Thereafter, Alyoshin [1], Suschansky [12], Grigorchuk [4] and Gupta–Sidki [5] gave various counter-examples.

We can tighten the above conjecture since we know that, $\circ(g)$ divides $\circ(G)$ the number of elements of the set of elements of G. Thus we can formulate,

Problem 2 (Ordinary Burnside Problem)**.** Let G be a finitely generated group for which there is a positive integer N such that for every element g we have $g^N = e$. Then is G finite?

(We call the smallest such integer N the *exponent* of G.) In 1968 Novikov and Adian [10] gave counter-examples for groups of odd exponents for the Ordinary Burnside Problem.

If we are primarily interested only in *finite* groups and their classification then we can again restrict the problem further. Thus Magnus [9] formulated the following problem.

Problem 3 (Restricted Burnside Problem)**.** Is there a finite number $A(k,N)$ of *finite* groups which are generated by k elements and have exponent N?

Alternatively one can ask if the order of all such groups is uniformly bounded. Hall and Higman [6] proved that the Restricted Burnside problem for a number N which can be factorised as $p_1^{n_1}\cdots p_r^{n_r}$ follows from the following three hypothesis:

(1) The Restricted Burnside Problem is true for $p_i^{n_i}$.
(2) There are at most finitely many finite simple group quotients which are k-generated and have exponent N.
(3) For each finite simple group quotient G as above the group of outer automorphisms of G is a solvable group.

Thus modulo the latter two problems which have to do with the Classification of Finite Simple groups, we reduced to a study of the Restricted Burnside Problem for p-groups. We note that a key step in the Classification of Finite Simple groups was the celebrated theorem of Feit and Thompson which won a Fields Medal in 1970. According to this theorem if N is odd then there are no finite simple groups in items (2) and (3) above.

3. p-Groups

For our purposes a p-group is a finite group such that it has p^a elements for some non-negative integer a. The Restricted Burnside problem for such groups can be stated as follows.

Problem 4 (Restricted Burnside Problem for p-groups)**.** Let G be a finite group with exponent p^n which is generated by k elements. Then G has p^a elements for some integer a. Is there a uniform bound $a(k,n)$ for a?

To study p-groups we first note that these are nilpotent. We define the Central series for G

$$G_1 = G \text{ and by induction on } i,\ G_i = [G, G_{i-1}]$$

Recall that G is *nilpotent* if G_i is the trivial group of order 1 for some i. Now if G is a p-group then the abelian groups G_i/G_{i+1} have order a power of p. Thus we can construct a finer series called the p-Central series for G a p-group

$$G_1 = G \text{ and } G_{i+1} \text{ is the subgroup generated by } [G, G_i] \text{ and the set } G_i^p$$

By the above discussion it follows that G_i becomes trivial for large enough i; in addition, each G_i/G_{i+1} is a vector space over $\mathbb{Z}/p\mathbb{Z}$ for all smaller i. The $\mathbb{Z}/p\mathbb{Z}$-vector space $L(G)$ is defined as

$$L(G) = \oplus_i G_i/G_{i+1}$$

The non-commutative structure of G can be caught by a Lie algebra structure on $L(G)$. We recall the definition of a Lie algebra.

Definition 1. A vector space L over a field k is said to be a *lie algebra* if there is a pairing $[,] : L \times L \to L$ with the following properties

$$[x, y] = -[y, x] \text{ and } [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

The Lie algebra structure on $L(G)$ is given by

$$G_i/G_{i+1} \times G_j/G_{j+1} \to G_{i+j}/G_{i+j+1}$$

where the map is $(\overline{x}, \overline{y}) \mapsto \overline{xyx^{-1}y^{-1}}$ (Check that this is well-defined!).

The above lie algebra has some additional structure. First is an identity proved by Higman [7]. If G has exponent p^n then

$$\sum_{\sigma \in S_{p^n-1}} \mathrm{ad}(a_{\sigma(1)}) \circ \mathrm{ad}(a_{\sigma(2)}) \circ \cdots \circ \mathrm{ad}(a_{\sigma(p^n-1)}) = 0$$

as a map $L(G) \to L(G)$; here $\mathrm{ad}(a) : L \to L$ for any element a in a Lie algebra is the map $b \mapsto [a, b]$.

The second identity is proved by Sanov [11]. Let x_i be the elements of $G_0/G_1 \subset L(G)$ corresponding to the finitely many generators g_i of G. Then for any ρ a commutator on the x_i we have $\mathrm{ad}(\rho)^{p^n} = 0$.

The main result of Zelmanov can formulated as follows.

Theorem 1 (Zelmanov)**.** *Let L be* any *Lie algebra over $\mathbb{Z}/p\mathbb{Z}$ which is generated as a Lie algebra by k elements x_i such that we have the Higman and Sanov identities.Then L is nilpotent as a Lie algebra.*

The interested reader can find this proof outlined in [13].

4. The Proof of the Problem

Now we claim that the Restricted Burnside Problem has an affirmative answer for exponent p^n. Let us examine this claim. To prove the Restricted Burnside Problem we need to show that the order p^a of a k-generated p-group G of exponent p^n is uniformly bounded by some constant $p^{a(k,n)}$. Now we have $\dim_{\mathbb{Z}/p\mathbb{Z}} L(G) = a$. Thus it is enough to bound the dimension of the Lie algebra $L(G)$. As in the case of the free group we can construct a *universal* Lie algebra L which is generated by k elements and satisfies the Higman and Sanov identities. Assuming the above theorem L is nilpotent. But then the abelian sub-quotients of the central series of L have a specified number of generators in terms of the generators of L and are thus finite dimensional. Thus L is itself finite dimensional, say of dimension $a(k, n)$. Since any $L(G)$ is a quotient of L its dimension is also bounded by $a(k, n)$ and this proves the result.

The rest of the Restricted Burnside Problem now follows since we have the result of Hall and Higman and also a complete Classification of Finite Simple groups by Feit, Thompson, Aschbacher et al.

Appendix A. A proof for Linear groups

Let G be a finitely generated subgroup of the group of invertible $n \times n$ matrices over complex numbers. We give a proof for the Ordinary Burnside problem first.

Let R be the collection of complex linear combinations of elements of G. The R is a finite dimensional vector space over the field of complex numbers spanned by the elements of G; thus there are elements $g_1, \ldots, g_r$ of G which form a basis of R.

Now suppose r is an element of R such that the traces $\mathrm{Trace}(g_i \cdot r) = 0$ all vanish. Then we obtain $\mathrm{Trace}(r^n) = 0$ for all positive integers n by expressing r^{n-1} as a linear combination of the g_i. But then these identities imply that $r = 0$. Thus an element g of G is *uniquely* determined once we know $\mathrm{Trace}(g_i \cdot g)$ for all i (if $\mathrm{Trace}(g_i \cdot g) = \mathrm{Trace}(g_i \cdot h)$ then apply the above argument to $r = g - h$).

Now we are given that each element of G satisfies $g^N = e$. Thus the trace of any element of G is a sum of n numbers of the form $\exp(2\pi\sqrt{-1} \cdot k/N)$ for $k = 1, \ldots, N$. But there are only finitely such sums. Thus by the previous paragraph there are only finitely many elements in G. (Exercise: use the above argument to provide an explicit bound).

We now show how to reduce the General Burnside Problem to the Ordinary Burnside Problem in this case. Let K be the field generated (over the field $\mathbb{Q}$ of rational numbers) by the matrix coefficients of the finite collection of generators of G. Let L be the subfield of K consisting of all algebraic numbers (elements satisfying a polynomial with rational coefficients). Since K is finitely generated L is a finite extension of $\mathbb{Q}$.

Now any element g of G has finite order. Hence the eigenvalues of g are roots of unity. Moreover, the characteristic polynomial of g has coefficients in the field K; since its roots are algebraic numbers the coefficients are in L. Thus the eigenvalues are roots of unity satisfying a polynomial of degree n over L; hence if d is the degree of the field extension L of $\mathbb{Q}$ we have roots of unity of degree at most $n \cdot d$ over $\mathbb{Q}$. There are only finitely many such roots of unity. Thus the order of G is bounded.

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