SOME SPECTRAL SEQUENCES FOR FILTERED COMPLEXES AND APPLICATIONS

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Abstract. We construct a heirarchy of spectral sequences for a filtered complex under a left-exact functor. As applications we prove (1) the existence of a Leray spectral sequence for de Rham cohomology, (2) the equivalence of this sequence with the "usual" Leray spectral sequence under the comparison isomorphism and (3) the isomorphism of the Bloch-Ogus spectral sequence with the Leray spectral sequence for the morphism from the fine site to the Zariski site.

1. Introduction

We construct a series of spectral sequences for the hypercohomology of a filtered complex. The basic constructions used are the shift (*dècalée*) operation of Deligne and its inverse [3].

Theorem 1.1. *Let $D : \mathcal{C} \to \mathcal{C}'$ be a left exact functor between abelian categories. Assume that $\mathcal{C}$ has enough injectives. For any good filtered complex (K, F) of objects in $\mathcal{C}$ we have natural spectral sequences for each $r \geq 1$.*

$$\mathrm{E}_r^{p,q} = \mathbb{D}^{p+q}(\mathcal{E}_{r-1,p,q}(K, F)) \implies \mathbb{D}^{p+q}(K)$$

For $r = 1$ this coincides with the spectral sequence for the hypercohomology of a filtered complex ([3]; section 1.4.5)*. For $r = 2$ and the trivial filtration F this coincides with the Leray spectral sequence for hypercohomology* (see [3]; section 1.4.7).

Here $\mathbb{D}^i$'s denote the hyperderived functors associated with D and $\mathcal{E}_{r-1,p,q}$ denotes the complex of $\mathcal{E}_{r-1}$ terms of the spectral sequence (see [3] section 1.3.1) for a filtered complex which contains $\mathcal{E}_r^{p,q}$ as the $(p+q)$-th term.

As applications, we provide proofs of some facts which are apparently well known to experts but are not well-documented in the literature (see however [7],[4]; the proof of the latter two applications is attributed to P. Deligne in [4] and [1]). The first is the existence of a Leray spectral sequence for de Rham cohomology. The second is the comparison of this spectral sequence with the Leray spectral sequence for singular cohomology. The third is the isomorphism of the Bloch–Ogus–Gersten spectral sequence with the Leray spectral sequence from E_2 onwards.

In Section 2 we recall some facts and definitions due to Deligne ([3] sections 1.3 and 1.4). In Section 3 we generalise some of these ideas to get the main results. The applications are elementary corollaries of the lemmas from section 3 and are

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proved in Section 4. In appendix A we recapitulate the required material from [3] and in appendix B we redo the main result using the language of derived categories.

I am grateful to V. Srinivas for advice and encouragement. I followed up his suggestion that the problem of the Bloch-Ogus spectral sequence and that of the Leray spectral sequence for de Rham cohomology could be related. J. Biswas and Balaji Srinivasan have had to listen to half-baked versions of the results and also had to correct my *wrong* impressions about the existence of connections for proper submersions.

2. Recapitulation some facts from [3]

We recall some facts proved by Deligne in ([3]; 1.3 and 1.4) with slightly different notations. The details can also be found in the appendix A. Let $\mathcal{C}$ be an abelian category. All objects, morphisms, etc. will be with reference to this category.

Definition 2.1. We say (K, F) is *good filtered complex in* $\mathcal{C}$ if K is a co-chain complex in $\mathcal{C}$ which is bounded below and F is a filtration on it which is compatible with the differential and is finite on K^n for each integer n.

We will always work with good filtered complexes in this paper. For any co-chain complex K let $\mathcal{H}^n(K)$ denote its n-th cohomology.

Fact 2.2 ([3]; 1.3.1)**.** For any good filtered complex (K, F) there is a spectral sequence

$$E_0^{p,n-p} = \mathcal{E}_0^{p,n-p}(K, F) = \operatorname{gr}_F^p K^n \implies \mathcal{H}^n(K)$$

such that the filtration induced by this spectral sequence on $\mathcal{H}^n(K)$ is the same as that induced by F.

Fact 2.3. The definitions of $\mathcal{E}_r^{p,q}(K, F)$ given in (*loc. cit.*) make sense for all integers r (not only positive integers r). The equalities $\mathcal{E}_r^{p,q} = \mathcal{E}_0^{p,q}$ for $r \leq 0$ hold and the differentials d_r are 0 for $r < 0$ due the compatibility of the filtration F with the differential on K.

Next Deligne defines various shifted filtrations associated with the given one. First of all we define ([3]; proof of 1.3.4)

$$\operatorname{Bac}(F)^p K^n := F^{p-n} K^n$$

Deligne shows that (*loc. cit.*),

$$\mathcal{E}_r^{p,n-p}(K, \operatorname{Bac}(F)) = \mathcal{E}_{r-1}^{p-n,2n-p}(K, F)$$

for all integers r. By induction on l we obtain

$$\mathcal{E}_r^{p,n-p}(K, \operatorname{Bac}^l(F)) = \mathcal{E}_{r-l}^{p-ln,(l+1)n-p}(K, F); \text{ for all integers } r \tag{1}$$

Moreover, Deligne notes the following fact about renumbering spectral sequences (actually he only notes it for s-r=1).

Fact 2.4. Let $E_a^{p,q}$ be the terms of a spectral sequence which starts at $a = r$, then we can obtain another spectral sequence ${E'}_b^{p,q}$ starting at the term $b = s$ by setting

$${E'}_b^{p,q} = E_{b-(s-r)}^{p-(s-r)(p+q),q+(s-r)(p+q)}$$

Next we consider the dual shifted filtration ([3]; section 1.3.5),

$$\mathrm{Dec}^*(F)^p K^n := \mathrm{im}(F^{p+n-1}K^{n-1} \to K^n) + F^{p+n}K^n = B_1^{p+n-1,1-p}(K,F)$$

Deligne shows (*loc. cit.*) that

$$\mathcal{E}_r^{p,n-p}(K, \mathrm{Dec}^*(F)) = \mathcal{E}_{r+1}^{p+n,-p}(K,F)$$

for all $r \geq 1$. By induction on l we obtain

(2) $$\mathcal{E}_r^{p,n-p}(K, (\mathrm{Dec}^*)^l(F)) = \mathcal{E}_{r+l}^{p+ln,(1-l)n-p}(K,F);\ \text{for all integers } r \geq 1.$$

Combining the results for Bac and Dec we see that

(3) $$\mathcal{E}_r^{p,n-p}(K, \mathrm{Bac}^l((\mathrm{Dec}^*)^l(F))) = \mathcal{E}_r^{p,n-p}(K,F);\ \text{for all } r \geq (l+1).$$

Definition 2.5 ([3]; 1.3.6)**.** A morphism $f : (K,F) \to (L,G)$ of good filtered complexes is said to be a *filtered quasi-isomorphism* if the morphisms $\mathrm{gr}^p(f) : \mathrm{gr}_F^p(K) \to \mathrm{gr}_G^p(L)$ is a quasi-isomorphism, i. e. $\mathcal{E}_1^{p,q}(f)$ are isomorpisms for all integers p and q.

Definition 2.6 ([3]; 1.4.5)**.** A *filtered injective resolution* of a good filtered complex (K,F) is a filtered quasi-isomorphism $(K,F) \to (L,G)$ such that the terms $\mathrm{gr}_L^p G^n$ are injective for all integers p and n.

A similar definition can be given with the property *injective* replaced by the property *D-acyclic* in the context of a left-exact functor $D : \mathcal{C} \to \mathcal{C}'$ as in ([3]; 1.4.1).

The following well-known fact is used in ([3]; 1.4.5)

Fact 2.7. If $\mathcal{C}$ has sufficiently many injectives then any good filtered complex (K,F) has a filtered injective resolution.

If $D : \mathcal{C} \to \mathcal{C}'$ is a left-exact functor and $\mathcal{C}$ has enough injectives, then we have the hypercohomologies $\mathbb{D}^i(K)$ in $\mathcal{C}'$ associated with any bounded below cochain complex K

Fact 2.8 ([3]; 1.4.4)**.** The $\mathbb{D}^i(K)$ are computed as the cohomologies of the complex $D(L)$ for any quasi-isomorphism $K \to L$ where the terms of L are D-acyclic.

We will also need the following well-known fact.

Fact 2.9. If K is a bounded below cochain complex such that $\mathcal{H}^p(K)$ are all D-acyclic then $\mathbb{D}^p(K) = D(\mathcal{H}^p(K))$.

3. Extensions to [3]

We now extend the definition ([3]; 1.3.6) slightly. Let us first introduce the notation $\mathcal{E}_{r,p,q}(K,F)$ for the complex of $\mathcal{E}_r$ terms which contains the term $\mathcal{E}_r^{p,q}(K,F)$ at the $p+q$-th position. When the integers p and q are irrelevant we will abbreviate this to $\mathcal{E}_r(K,F)$. Note that we have the equality of complexes $\mathcal{E}_{r,p,q} = \mathcal{E}_{r,p+r,q-r+1}$.

Definition 3.1. A morphism $f : (K,F) \to (L,G)$ of good filtered complexes is said to be a *level-r filtered quasi-isomorphism* if the morphisms $\mathcal{E}_{r-1}(f)$ induce quasi-isomorphisms $\mathcal{E}_{r-1}(K,F) \to \mathcal{E}_{r-1}(L,G)$, i. e. $\mathcal{E}_r^{p,q}(f)$ are isomorpisms for all integers p and q.

A level-1 filtered quasi-isomorphism is what was earlier (2.5) called a filtered quasi-isomorphism.

We extend the definition ([3]; 1.4.5) in a similar way.

Definition 3.2. A *level-r filtered injective resolution* of a good filtered complex (K, F) is a level-r filtered quasi-isomorphism $(K, F) \to (L, G)$ such that the terms $\mathcal{E}_{r'}^{p,q}(L, G)$ are injective for all $r' < r$ and all integers p and q.

A similar definition can be given with the property *injective* replaced by the property *D-acyclic* in the context of a left-exact functor $\mathcal{C} \to \mathcal{C}'$ as before. A level-1 filtered injective resolution is what was earlier (2.6) called a filtered injective resolution.

Example 3.3. Let K be any complex on objects in $\mathcal{C}$. We put the *trivial filtration* F on K by defining $F^0K = K$ and $F^1K = 0$. Then we note as in ([3]; 1.4.6) that

$$\mathrm{Dec}^*(F)^p K^n = \begin{cases} 0 & \text{if } n > 1-p \\ d(K^{-p}) & \text{if } n = 1-p \\ K^n & \text{if } n < 1-p \end{cases}$$

Thus, $\mathrm{gr}^p_{\mathrm{Dec}^*(F)} K$ is the complex concentrated in degrees $-p$ and $1-p$.

$$K^{-p}/d(K^{-p-1}) \to d(K^{-p})$$

There is a natural morphism to this from the single term complex $\mathcal{H}^{-p}(K)$ concentrated in degree $-p$ which is clearly a quasi-isomorphism.

Fact 3.4. Let L denote the total complex of a Cartan–Eilenberg resolution [2] I of K. Let $G^p(L)$ be the total complex of the subcomplex $I^{\geq p}$. Let F be the trivial filtration on K. Then the natural morphism $(K, F) \to (L, G)$ is a level-2 filtered injective resolution.

Proof. As noted above we have

$$\mathcal{E}_2^{0,n}(K, F) = \mathcal{E}_1^{-n,2n}(K, \mathrm{Dec}^*(F)) = \mathcal{H}^n(K)$$

and the remaining $\mathcal{E}_2$ terms are 0. We have the identity $\mathcal{E}_0^{p,n}(J, G) = I^{p,n}$ and so we deduce $\mathcal{E}_1^{p,n}(J, G) = \mathcal{H}^n(I^p)$. Since I is a Cartan-Eilenberg resolution these $\mathcal{E}_1$ terms give an injective resolution of $\mathcal{H}^n(K)$. Thus we have the result. □

We use the fact (2.7) to prove

Lemma 3.5. *If $\mathcal{C}$ has sufficiently many injectives then any good filtered complex (K, F) has a level-r injective resolution for any $r \geq 1$.*

Proof. Let $(K, (\mathrm{Dec}^*)^{r-1}(F)) \to (L, G)$ be a (level-1) filtered injective resolution (which exists by (2.7)). Consider the composite morphism

$$(K, F) \to (K, \mathrm{Bac}^{r-1}(\mathrm{Dec}^*)^{r-1}(F)) \to (L, \mathrm{Bac}^{r-1}(G))$$

By (3) we see that the first morphism is a level-r quasi-isomorphism. Also by (1) we see that the second morphism is a level-r quasi-isomorphism. Hence the composite is also a level-r quasi-isomorphism. Now by (1) and (2.3) we have for $r' < r$

$$\mathcal{E}_{r'}(L, \mathrm{Bac}^{r-1}(G)) = \mathcal{E}_{r'-r+1}(L, G) = \mathcal{E}_0(L, G)$$

By assumption the latter terms are injective. □

Next we note the naturality of such a resolution.

Lemma 3.6. *Suppose (K,F) is a good filtered complex such that $\mathcal{E}_{r'}^{p,q}(K,F)$ are injective for all $r' \leq r$ and all integers p and q. Let $f:(K,F)\to(L,G)$ be a level-r injective resolution then there is a morphism $g:(L,G)\to(K,F)$ such that $\mathcal{E}_{r-1}(f\circ g)$ and $\mathcal{E}_{r-1}(g\circ f)$ are homotopic to identity.*

Proof. Note that $\mathcal{E}_{r-1}(f):\mathcal{E}_{r-1}(K,F)\to\mathcal{E}_{r-1}(L,G)$ is a quasi-isomorphism of complexes of injectives. Hence there is a morphism $g_{r-1}:\mathcal{E}_{r-1}(L,G)\to\mathcal{E}_{r-1}(K,F)$ such that $\mathcal{E}_{r-1}(f)\circ g_{r-1}$ and $g_{r-1}\circ\mathcal{E}_{r-1}(f)$ are homotopic to identity. By induction we assume that we are given the morphism $g_{r'}:\mathcal{E}_{r'}(L,G)\to\mathcal{E}_{r'}(K,F)$. We wish to find a morphism $g_{r'-1}:\mathcal{E}_{r'-1}(L,G)\to\mathcal{E}_{r'-1}(K,F)$ such that it induces $g_{r'}$ on the cohomology of the $\mathcal{E}_{r'-1}$ terms (which is $\mathcal{E}_{r'}$). This is possible since the $\mathcal{E}_{r'-1,p,q}$'s are bounded below complexes of injectives. Thus we obtain $g_{0,p}:\operatorname{gr}_G^p L\to\operatorname{gr}_F^p K$. Again we have that K and L are bounded below complexes of injectives and so we obtain the required morphism g which satisfies $\operatorname{gr}^p(g)=g_{0,p}$. ☐

We have the following modification of ([3]; 1.4.5):

Lemma 3.7. *Let $D:\mathcal{C}\to\mathcal{C}'$ be a left-exact functor and assume that $\mathcal{C}$ has enough injectives. Let $\mathbb{D}^i$ denote the associated hypercohomology functors and let (K,F) be a good filtered complex in $\mathcal{C}$. Then for any $r\geq 1$ we have a natural spectral sequence*

$$E_r^{p,q}=\mathbb{D}^{p+q}(\mathcal{E}_{r-1,p,q}) \implies \mathbb{D}^{p+q}(K)$$

Proof. Let $(K,F)\to(L,G)$ be a level-r filtered D-acyclic resolution (for example we can take a level-r filtered injective resolution by lemma (3.5)). Consider the good filtered complex $(D(L),D(G))$ in $\mathcal{C}'$. The associated spectral sequence is

$$E_0^{p,q}=\operatorname{gr}_{D(G)}^p D(L)^{p+q} \implies \mathcal{H}^{p+q}(D(L))$$

Now, by (2.8) the latter term is $\mathbb{D}^{p+q}(K)$. Since $\operatorname{gr}_G^p(L^n)$ are D-acyclic for all integers p and n and the filtrations are finite, we see that $\operatorname{gr}_{D(G)}^p(D(L))=D(\operatorname{gr}_G^p(L))$.

Now by definition $E_l^{p,q}=\mathcal{H}^{p+q}(E_{l-1,p,q})$. Thus we obtain

$$E_1^{p,q}=\mathcal{H}^{p+q}(D(\operatorname{gr}_G^p(L)))=\mathbb{D}^{p+q}(\operatorname{gr}_G^p(L))$$

since $\operatorname{gr}_G^p(L^n)$ are D-acyclic. We now claim by induction that

$$E_{r'}^{p,q}=\mathbb{D}^{p+q}(\mathcal{E}_{r'-1,p,q}(L,G));\text{for } r'\leq r$$

Assume this for $r'-1$. Now since $r'-1<r$ we have $\mathcal{E}_{r'-1}(L,G)$ consists of D-acyclics. Thus we see by (2.9) that

$$E_{r'-1}^{p,q}=\mathbb{D}^{p+q}(\mathcal{E}_{r'-2,p,q}(L,G))=D(\mathcal{H}^{p+q}(\mathcal{E}_{r'-2,p,q}(L,G)))=D(\mathcal{E}_{r'-1}^{p,q})$$

But then by definition of $\mathbb{D}^i$'s we have

$$E_{r'}^{p,q}=\mathcal{H}^{p+q}(E_{r'-1,p,q})=\mathcal{H}^{p+q}(D(\mathcal{E}_{r'-1,p,q}(L,G)))=\mathbb{D}^{p+q}(\mathcal{E}_{r'-1,p,q})$$

This proves the claim by induction.

Now we have $\mathcal{E}_{r-1,p,q}(K,F)\to\mathcal{E}_{r-1,p,q}(L,G)$ is an D-acyclic resolution. Thus

$$\mathbb{D}^{p+q}(\mathcal{E}_{r-1,p,q}(K,F))=\mathbb{D}^{p+q}(\mathcal{E}_{r-1,p,q}(L,G))$$

Hence we obtain the required spectral sequence. The naturality of this spectral sequence easily follows from the lemma (3.6) by the usual techniques. ☐

Now we note that for $r = 1$ this spectral sequence is exactly the one constructed by Deligne in ([3];1.4.5). For $r = 2$ we see that this is the Leray spectral sequence for hypercohomology by applying the level-2 injective resolution given by Cartan-Eilenberg (3.4). From the above proof we see that we obtain our "new" spectral sequence. On the other hand the E_2 spectral sequence associated with the Cartan-Eilenberg resolution is precisely what is called the Leray spectral sequence for hypercohomology. This completes the proof of the main theorem (1.1).

4. Applications

We have the following corollary of Lemma (3.7).

Corollary 4.1. *Let $X \to S$ be a proper smooth morphism of varieties. Then there is a spectral sequence*

$$\mathrm{E}_2^{p,q} = \mathbb{H}^p(S, \Omega_S^* \otimes \mathrm{R}^q_{dR}(X/S)) \implies \mathrm{H}^{p+q}_{dR}(X)$$

Here, the complex $\Omega_S^* \otimes \mathrm{R}^q_{dR}(X/S)$ is the one arising from the Gauss–Manin connection.

Proof. Let $\Omega_X^* \to \mathcal{K}'$ be the Godement resolution of sheaves of abelian groups on X. The direct image $\mathcal{K}$ of $\mathcal{K}'$ is a differential graded module for the sheaf Ω_S^* of diferential forms on S. The irrelevant ideal $\Omega_S^{\geq 1}$ induces an ideal theoretic filtration F on $\mathcal{K}$. Since the morphism $X \to S$ is proper and smooth we see that

$$\mathcal{E}_1^{p,q}(\mathcal{K}, F) = \Omega_S^p \otimes \mathrm{R}^q_{\mathrm{dR}}(X/S)$$

Furthermore, the d_1 differential of this sequence can be identified with the morphism arising out of the Gauss–Manin connection (see [5]). Now by applying the lemma (3.7) we have the required spectral sequence. □

We have the following corollaries of Theorem (1.1).

Corollary 4.2. *The Leray spectral sequence for a proper submersion of smooth (C^∞) manifolds coincides with the E_2-spectral sequence arising out of the Gauss-Manin local system from Lemma (3.7).*

Proof. Let $f : X \to S$ be a proper submersion of C^∞-manifolds. Let $\mathcal{A}_X$ (resp. $\mathcal{A}_S$) be the sheaf of differential forms on X (resp. S). These are sheaves of differential graded algebras and $f_*(\mathcal{A}_X)$ is in addition a sheaf of differential graded modules for $\mathcal{A}_S$. The irrelevant ideal $\mathcal{A}_S^{\geq 1}$ thus gives rise to a filtration F on $f_*(\mathcal{A}_X)$. The $\mathcal{E}_1$ terms for the spectral sequence (2.2) for this complex give us a complex

$$0 \to \mathcal{H}^q(f_*(\mathcal{A}_X)) \to \mathcal{E}_1^{0,q}(f_*(\mathcal{A}_X), F) \to \mathcal{E}_1^{1,q}(f_*(\mathcal{A}_X), F) \to \cdots$$

Now as above we can show that

$$\mathcal{E}_1^{p,q}(f_*(\mathcal{A}_X, F) = \mathcal{A}_S^p \otimes_{\mathbb{C}_S} \mathrm{R}^q\, f_*(\mathbb{C}_X)$$

And the d_1 differential can be identified with the morphism arising out of the Gauss-Manin connection on the vector bundle associated with the local system $\mathrm{R}^q\, f_*(\mathbb{C}_X)$. But then the above complex becomes an *exact* sequence

$$0 \to \mathrm{R}^q\, f_*(\mathbb{C}_X) \to \mathcal{A}_S^0 \otimes_{\mathbb{C}_S} \mathrm{R}^q\, f_*(\mathbb{C}_X) \to \mathcal{A}_S^1 \otimes_{\mathbb{C}_S} \mathrm{R}^q\, f_*(\mathbb{C}_X) \to \cdots$$

Now the sheaves $\mathcal{A}_S^p$ are fine and hence are acyclic for the functor of global sections. Thus if G denotes the trivial filtration on $f_*(\mathcal{A}_X)$ we have a level-2 $\Gamma(S, \cdot)$-acyclic resolution $(f_*(\mathcal{A}_X), G) \to (f_*(\mathcal{A}_X), F)$. We can apply the theorem (1.1) to conclude that the two E_2 spectral sequences coincide by naturality. □

The algebro-geometric version of the above uses the regularity of the Gauss-Manin connection [4]. Note that the latter spectral sequence has a purely algebraic construction as in (4.1).

Corollary 4.3. *The Leray spectral sequence for a proper submersion of complex algebraic manifolds coincides with the E_2-spectral sequence arising out of the Gauss-Manin connection from Lemma (3.7).*

Proof. Let $f_*(\mathcal{A}_X)$ be the complex with the natural filtrations as in the proof of the previous corollary. By the Poincaré lemma we have a quasi-isomorphism $\Omega_X \to \mathcal{A}_X$ of complex on X. Thus we have a quasi-isomorphism $i : Rf_*(\Omega_X) \to f_*(\mathcal{A}_X)$. The former is a sheaf of differential graded algebras which is a differential graded module for Ω_S. Thus we have a filtration on $Rf_*(\Omega_X)$ induced by the irrelevant ideal $\Omega_S^{\geq 1}$. This makes the above morphism i a morphism of filtered complexes on S. This gives a morphism of spectral sequences constructed using Lemma (3.7):

$$\mathrm{E}_2^{p,q}(\Omega) = \mathbb{H}^p(S, \Omega_S^* \otimes \mathrm{R}^q_{\mathrm{dR}}(X/S)) \to \mathrm{E}_2^{p,q}(\mathcal{A}) = \mathbb{H}^p(S, \mathcal{A}_S^* \otimes \mathrm{R}^q f_*(\mathbb{C}_X))$$

By the regularity of the Gauss-Manin system this is an isomorphism. Now we combine this with the previous corollary to obtain the result. □

Corollary 4.4. *The Bloch-Ogus spectral sequence for any Poincaré duality theory coincides with the Leray spectral sequence for the morphism from the* fine *site to the Zariski site.*

Proof. We will use the exactness of the Gersten complex as proved by Bloch and Ogus [1]. Let Y denote a *fine* site associated with a variety X. Let $\mathcal{K}'$ be a complex of injective sheaves which computes the cohomology of Y in a suitable Poincaré duality theory. Let $\mathcal{K}$ be the resulting complex of sheaves on X obtained by taking direct image. Then for any Zariski open set U in X the global sections $\mathcal{K}(U)$ give a complex that computes the cohomology on the fine site associated with U. We have a natural filtration F on $\mathcal{K}$ by the codimension of support.

Let Z be a subset of U which is of pure codimension p and $W \subset Z$ be a subset which is pure of codimension $p+1$ in U. Then we have the complexes $\mathcal{K}_Z(U) = \ker(\mathcal{K}(U) \to \mathcal{K}(U-Z)$ and $\mathcal{K}_W(U) = \ker(\mathcal{K}(U) \to \mathcal{K}(U-W)$ which compute the cohomology of U with supports in Z and W respectively. We see that the quotient complex $\mathcal{K}_Z(U)/\mathcal{K}_W(U)$ is naturally isomorphic to $\mathcal{K}_{Z-W}(U-W) = \ker(\mathcal{K}(U-W) \to \mathcal{K}(U-Z))$. Now as we take direct limits over pairs (Z,W) we obtain $F^p\mathcal{K}(U) = \varinjlim \mathcal{K}_Z(U)$ and $F^{p+1}\mathcal{K} = \varinjlim \mathcal{K}_W(U)$. Furthermore, we see that the cohomology of the complex $\mathrm{gr}_F^p \mathcal{K}$ at the q-th place is the term $\oplus_{x\in X^p}(i_x)_* \mathrm{H}^q(k(x))$ which is a flasque sheaf and hence in particular is $\Gamma(X,\cdot)$-acyclic. Thus if G denotes the trivial filtration on $\mathcal{K}$ we have a level-2 acyclic resolution $\mathcal{K}, G) \to (\mathcal{K}, F)$ by applying the result of Bloch and Ogus (*loc. cit.* ; Theorem 4.2). Thus we see that the conditions of the theorem (1.1) are satisfied and the two spectral sequences coincide. □

Appendix A. Spectral sequences and filtered complexes

We reproduce some facts proved by Deligne in ([3]; 1.3 and 1.4) with slightly different notations. Let (K,F) be a filtered co-chain complex which is bounded below and such that the filtration is finite on each term of the complex. Moreover, we assume that the differential on K is compatible with the filtration. We define for each integer r,

$$\begin{aligned}
\mathcal{Z}_r^{p,n-p}(K,F) &:= \ker(F^pK^n \to K^{n+1}/F^{p+r}K^{n+1}) \\
\mathcal{B}_r^{p,n-p}(K,F) &:= \mathrm{im}(F^{p-r+1}K^{n-1} \to K^n) + F^{p+1}K^n \\
\mathcal{E}_r^{p,n-p}(K,F) &:= \mathcal{Z}_r^{p,n-p}/(\mathcal{Z}_r^{p,n-p} \cap \mathcal{B}_r^{p,n-p})
\end{aligned}$$

Note that $\mathcal{E}_r = \mathcal{E}_0$ for all $r \leq 0$ because the differential is compatible with the filtration. One easily shows that the $\mathcal{E}_r^{p,q}$'s are the terms of a spectral sequence

$$E_0^{p,n-p} = \mathcal{E}_0^{p,n-p}(K,F) \implies \mathcal{H}^n(K)$$

such that the filtration induced by this spectral sequence on $\mathcal{H}^n(K)$ is the same as that induced by F.

Next we define various shifted filtrations associated with the given one. First of all we define

$$\mathrm{Bac}(F)^pK^n := F^{p-n}K^n$$

One then computes that for all integers r,

$$\begin{aligned}
\mathcal{Z}_r^{p,n-p}(K,\mathrm{Bac}(F)) &= \ker(F^{p-n}K^n \to K^{n+1}/F^{p+r-n-1}K^{n+1}) \\
&= \mathcal{Z}_{r-1}^{p-n,2n-p}(K,F) \\
\mathcal{B}_r^{p,n-p}(K,\mathrm{Bac}(F)) &= \mathrm{im}(F^{p-r-n+2}K^{n-1} \to K^n) + F^{p+1-n}K^n \\
&= \mathcal{B}_{r-1}^{p-n,2n-p}(K,F) \\
\mathcal{E}_r^{p,n-p}(K,\mathrm{Bac}(F)) &= \mathcal{E}_{r-1}^{p-n,2n-p}(K,F)
\end{aligned}$$

In particular, we see that $\mathcal{E}_r(K,\mathrm{Bac}(F)) = \mathcal{E}_1(K,\mathrm{Bac}(F)) = \mathcal{E}_0(K,F)$ for all $r \leq 1$.

Next we consider the dual shifted filtration,

$$\mathrm{Dec}^*(F)^pK^n := \mathrm{im}(F^{p+n-1}K^{n-1} \to K^n) + F^{p+n}K^n = B_1^{p+n-1,1-p}(K,F)$$

One computes the following equations for all $r \geq 0$,

$$\begin{aligned}
\mathcal{Z}_r^{p,n-p}(K,\mathrm{Dec}^*(F)) &= \ker(B_1^{p+n-1,1-p}(K,F) \to K^{n+1}/B_1^{p+r+n,1-r-p}(K,F)) \\
&= \mathrm{im}(F^{p+n-1}K^{n-1} \to K^n) + \mathcal{Z}_{r+1}^{p+n,-p}(K,F) \\
\mathcal{B}_r^{p,n-p}(K,\mathrm{Dec}^*(F)) &= \mathrm{im}(B_1^{p+n-r-1,r-p}(K,F) \to K^n) + B_1^{p+n,-p}(K,F) \\
&= \mathcal{B}_{r+1}^{p+n,-p}(K,F)
\end{aligned}$$

Now for $r \geq 1$ we have $F^{p+n+r}K^n \subset \mathcal{B}_{r+1}^{p+n,-p}$. Hence one deduces that

$$\mathcal{E}_r^{p,n-p}(K,\mathrm{Dec}^*(F)) = \mathcal{E}_{r+1}^{p+n,-p}(K,F)$$

for all $r \geq 1$.

Appendix B. Filtered derived categories

We re-prove the main results using the language of derived categories ([6]).

Let $\mathcal{C}$ be an abelian category. Let $\mathcal{FK}$ denote the category whose objects are pairs (K,F) where K is a cochain complex with terms in $\mathcal{C}$ and F is a filtration on K such that K is bounded below and the filtration F is finite on each term of K. Moreover, we assume that the differential on K is compatible with the filtration. The group $Hom_{\mathcal{FK}}((K,F),(L,G))$ is the group of morphisms of complexes compatible with the filtration modulo the subgroup of homotopically trival morphisms. We call

$\mathcal{FK}$ the Homotopy category of filtered complexes in $\mathcal{C}$. One checks that this is a triangulated category.

A morphism $f : (K, F) \to (L, G)$ in $\mathcal{FK}$ is called a filtered quasi-isomorphism if it induces quasi-isomorphisms $\mathrm{gr}^p_F(K) \to \mathrm{gr}^p_G(L)$ for every p. It is well known that filtered quasi-isomorphisms form a saturated multiplicatively closed set and hence we can form the localised category $\mathcal{DF}$ with a functor $\mathcal{FK} \to \mathcal{DF}$ which is universal for the property that all filtered quasi-isomorphisms become isomorphisms under this functor.

In section (3) we defined level-r filtered quasi-isomorphisms and the Dec shift operation on filtrations. The result of ([3]; section 1.3.4) can then be restated as follows:

Lemma B.1 (Deligne)**.** *The functor* $\mathrm{Dec} : \mathcal{FK} \to \mathcal{FK}$ *carries level-r filtered quasi-isomorphisms to level-$(r-1)$ filtered quasi-isomorphisms for all $r \geq 2$. Conversely, if* $\mathrm{Dec}(f)$ *is a level-$(r-1)$ filtered quasi-isomorphism then f is a level-r filtered quasi-isomorphism.*

Let us apply this to the composite functor

$$\mathcal{FK} \xrightarrow{\mathrm{Dec}^l} \mathcal{FK} \to \mathcal{DF}$$

By the lemma we see that the set of level-$(l+1)$ filtered quasi-isomorphims is precisely the set of morphisms that become isomorphisms under the composite. Since Dec is clearly a triangulated functor we can apply the results of [6] to conclude that level-$(l+1)$ filtered quasi-isomorphisms form a saturated multiplicatively closed set. Hence we can form the quotient category of $\mathcal{FK}$ by inverting such morphisms. We denote this category by $\mathcal{DF}_{l+1}$. Note that $\mathcal{DF}$ and $\mathcal{DF}_1$ are identical.

The operation Bac on filtrations also gives a functor $\mathrm{Bac} : \mathcal{FK} \to \mathcal{FK}$ which carries level-r filtered quasi-isomorphmisms to level-$r+1$ filtered quasi-isomorphisms. It thus induces a functor $\mathrm{Bac} : \mathcal{DF}_r \to \mathcal{DF}_{r+1}$.

Let $\mathcal{D}$ denote the derived category of bounded below co-chain complexes with terms in $\mathcal{C}$. We have natural forgetful functors $\mathcal{DF}_r \to \mathcal{D}$ by forgetting the filtrations (note that a level-r filtered quasi-isomorphism is in particular a quasi-isomorphism of the underlying complexes). We also have for each integer p a functor $\mathrm{gr}^p : \mathcal{DF} \to \mathcal{D}$. More generally, for each integer $r \geq 1$ and each p we have $\mathcal{E}_{r-1,p,0} : \mathcal{DF}_r \to \mathcal{D}$.

Let $T : \mathcal{C} \to \mathcal{A}$ be a left-exact functor with values in an abelian category $\mathcal{A}$. Moreover, let us assume that $\mathcal{C}$ has sufficiently many injectives. Then we have the hypercohomology functors $RT^i : \mathcal{D} \to \mathcal{A}$. The spectral sequence for the hypercohomology of a filtered complex is then

$$E_1^{p,q} = RT^{p+q}(\mathrm{gr}^p_F K) \implies RT^{p+q}(K)$$

which is naturally associated with any element (K, F) of $\mathcal{DF}$. We can also write the E_1 terms as $E_1^{p,n-p} = RT^n(\mathcal{E}_{0,p,n-p}(K, F))$.

We then construct a level-r spectral sequence for the hypercohomology of a filtered complex

$$E_r^{p,n-p} = RT^n(\mathcal{E}_{r-1,p,n-p}(K, F)) \implies RT^n(K)$$

One way to construct such a sequence is as follows. We have a natural spectral sequence

$$E_1^{p,n-p} = RT^n(\mathcal{E}_{0,p,n-p}(K, (\mathrm{Dec}^*)^{r-1}(F))) \implies RT^n(K)$$

But we use (2.4) to re-write this as an E_r spectral sequence by setting

$$E_r^{p,n-p} = RT^n(\mathcal{E}_{0,p+(r-1)n,(2-r)n-p}(K, (\mathrm{Dec}^*)^{r-1}(F))) \\ = RT^n(\mathcal{E}_{r,p,n-p}(K, \mathrm{Bac}^{r-1}(\mathrm{Dec}^*)^{r-1}(F)))$$

Now the natural morphism $(K, F) \to (K, \mathrm{Bac}^{r-1}(\mathrm{Dec}^*)^{r-1}(F))$ is a level-r quasi-isomorphism in $\mathcal{DF}_r$ and so we have

$$RT^n(\mathcal{E}_{r-1,p,n-p}(K, F)) \to RT^n(\mathcal{E}_{r-1,p,n-p}(K, \mathrm{Bac}^{r-1}(\mathrm{Dec}^*)^{r-1}(F)))$$

is an isomorphism. This gives us the required spectral-sequence which is natural for elements of $\mathcal{DF}_r$.

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