THE ASYMMETRIC TOP

We will "approximate" the asymmetric top by a finite set of point masses which are attached to each other by massless, rigid rods which ensure that the motion of the whole ensemble is described by a path in the group of Euclidean motions in space. The exposition given below is similar to that which can be found in the book on Classical Mechanics by V. I. Arnold.

For the i-th point mass with mass m_i, its position $\vec{r}_i$ is a function of time t given by the formula

$$\vec{r}_i(t) = g(t) \cdot \vec{v}_i + \vec{r}(t)$$

where $\vec{v}_i$ is a constant vector and g is a path in the orthogonal group. Let $M = \sum_i m_i$ denote the total mass. The position $\vec{R}$ of the centre of mass is given by

$$\vec{R} = \sum_i m_i \vec{r}_i / M.$$

Moreover, if $\vec{V} = \sum_i m_i \vec{v}_i / M$, then we have

$$\vec{R}(t) = g(t) \cdot \vec{V} + \vec{r}(t).$$

If we set $\vec{u}_i = \vec{v}_i - \vec{V}$, then we obtain

$$\begin{aligned}\vec{r}_i(t) &= g(t) \cdot \vec{u}_i + \vec{R}(t) \\ &= \text{ motion } \textit{about} \text{ the centre of mass } + \text{ motion } \textit{of} \text{ the centre of mass}\end{aligned}$$

We note that $\sum_i m_i \vec{u}_i = \vec{0}$.

Now let us consider the velocity vectors of individual point masses,

$$\dot{\vec{r}}_i(t) = \dot{g}(t) \cdot \vec{u}_i + \dot{\vec{R}}(t)$$

Since $g(t)$ is a path in the space of orthogonal matrices, we see that $g(t)^{-1}\dot{g}(t)$ is a skew-symmetric matrix. Hence there is a vector valued function $\vec{\Omega}(t)$ so that for any vector $\vec{u}$ we have

$$\vec{\Omega}(t) \times \vec{u} = g(t)^{-1} \cdot \dot{g}(t) \cdot \vec{u}$$

$\vec{\Omega}(t)$ is called the *angular velocity* about the centre of mass in the *body frame.* If $\vec{\omega} = g(t) \cdot \vec{\Omega}$, then by applying $g(t)$ on all terms of the above equation, we see that for any vector $\vec{r}$ we have

$$\vec{\omega}(t) \times \vec{r} = \dot{g}(t) \cdot g(t)^{-1} \cdot \vec{r}$$

$\vec{\omega}(t)$ is called the angular velocity about the centre of mass in the *stationary frame.* Thus, we can re-write the velocity vector of the i-th point mass

$$\dot{\vec{r}}_i(t) = \vec{\omega}(t) \times (\vec{r}_i - \vec{R}(t)) + \dot{\vec{R}}(t)$$

The momentum of the system as a whole is given by

$$\vec{p} = \sum_i m_i \dot{\vec{r}}_i = M\dot{\vec{R}}$$

The angular momentum of the system as a whole in the *stationary* co-ordinates is given by

$$\vec{\lambda}_{\text{tot}} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i = \sum_i m_i(\vec{r}_i - \vec{R}) \times \dot{\vec{r}}_i + \sum_i m_i \vec{R} \times \dot{\vec{r}}_i$$
$$= \sum_i m_i(\vec{r}_i - \vec{R}) \times (\vec{\omega} \times (\vec{r}_i - \vec{R}(t))) + M\vec{R} \times \dot{\vec{R}}$$

(the remaining terms vanish because $\sum_i m_i \vec{r}_i = M\vec{R}$). The latter term in the above expression for $\vec{\lambda}_{\text{tot}}$ has the obvious interpretation as the angular momentum of (a point mass concentrated at) the centre of mass. The former term

$$\vec{\lambda} = \sum_i m_i(\vec{r}_i - \vec{R}) \times (\vec{\omega}(t) \times (\vec{r}_i - \vec{R}))$$

is referred to as the angular momentum of the system *about* the centre of mass in the *stationary frame*. Applying $g(t)^{-1}$ to the entire expression gives

$$\vec{\Lambda} = \sum_i m_i \vec{u}_i \times (\vec{\Omega} \times \vec{u}_i)$$

which is called the angular momentum in the *body frame*. We note that the map

$$\vec{w} \mapsto I(\vec{w}) = \sum_i m_i \vec{u}_i \times (\vec{w} \times \vec{u}_i)$$

depends only on the initial position of the point masses with respect to their centre of mass and is thus associated to the configuration or "shape" of this system; I is called the *moment of inertia* or more strictly, the moment of inertia *tensor* of the configuration of the point masses.

We can also compute the total kinetic energy T_{tot} of the system

$$T_{\text{tot}} = \tfrac{1}{2}\sum_i m_i ||\dot{\vec{r}}_i||^2 = \tfrac{1}{2}\sum_i m_i(\dot{\vec{r}}_i - \dot{\vec{R}}).\dot{\vec{r}}_i + \tfrac{1}{2}\sum_i m_i \dot{\vec{R}}.\dot{\vec{r}}_i$$
$$= \tfrac{1}{2}\sum_i m_i(\dot{\vec{r}}_i - \dot{\vec{R}}).(\dot{\vec{r}}_i - \dot{\vec{R}}) + \tfrac{1}{2}M\dot{\vec{R}}.\dot{\vec{R}}$$
$$= \tfrac{1}{2}\sum_i m_i ||\vec{\omega} \times (\vec{r}_i - \vec{R})||^2 + \tfrac{1}{2}M||\dot{\vec{R}}||^2$$

The latter term is the kinetic energy of a point mass concentrated at the centre of mass. The former term

$$T = \tfrac{1}{2}\sum_i m_i ||\vec{\omega} \times (\vec{r}_i - \vec{R})||^2 = \tfrac{1}{2}\sum_i m_i ||\vec{\Omega} \times \vec{u}_i||^2$$

is called the kinetic energy *about* the centre of mass or the *rotational* kinetic energy. Note that

$$T = \tfrac{1}{2}\sum_i m_i \vec{\Omega}.(\vec{u}_i \times (\vec{\Omega} \times \vec{u}_i)) = \tfrac{1}{2}\vec{\Omega}.I(\vec{\Omega})$$

Using the identity

$$\vec{x}.(\vec{y} \times \vec{z}) = \vec{y}.(\vec{z} \times \vec{x})$$

we obtain $\vec{x}.I(\vec{w}) = \vec{w}.I(\vec{x})$; in other words I is given by a symmetric matrix. We also see that

$$\vec{w}.I(\vec{w}) = \sum_i m_i ||(\vec{w} \times \vec{u}_i)||^2 = ||\vec{w}||^2 \sum_i m_i \text{ (distance of } \vec{u}_i \text{ from } \mathbb{R}\vec{w})^2$$

Thus for a non-zero vector $\vec{w}$, the latter quantity

$$I_{\vec{w}} = \sum_i m_i \text{ (distance of } \vec{u}_i \text{ from } \mathbb{R}\vec{w})^2$$

is sometimes called the moment of inertia in the direction $\vec{w}$; it does not depend on the magnitude of $\vec{w}$. Because I is symmetric, there is an orthonormal eigen-basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ for I. The numbers $I_i = I_{\vec{e}_i}$ are called the principal moments of the system.

The behaviour of the i-th point mass is dependent on the force $\vec{F}_i$ acting on it. Newton's law states that $\vec{F}_i = m_i\ddot{\vec{r}}_i$. From the expression for $\dot{\vec{r}}_i$ and the identity $\dot{\vec{\omega}} = g \cdot \dot{\vec{\Omega}}$ we obtain

$$\begin{aligned}\ddot{\vec{r}}_i &= \dot{g} \cdot (\vec{\Omega} \times \vec{u}_i) + g \cdot (\dot{\vec{\Omega}} \times \vec{u}_i) + \ddot{\vec{R}} \\ &= g \cdot (\vec{\Omega} \times (\vec{\Omega} \times \vec{u}_i)) + g \cdot (\dot{\vec{\Omega}} \times \vec{u}_i) + \ddot{\vec{R}} \\ &= \vec{\omega} \times (\vec{\omega} \times (\vec{r}_i - \vec{R})) + \dot{\vec{\omega}} \times (\vec{r}_i - \vec{R}) + \ddot{\vec{R}}\end{aligned}$$

Let $\vec{F} = \sum_i \vec{F}_i$. Then we have $\vec{F} = M\ddot{\vec{R}}$ so that the centre of mass of the system behaves as if the force $\vec{F}$ is acting on (a point mass centred at) it. Now we compute the rate of change of angular momentum,

$$\begin{aligned}\dot{\vec{\lambda}}_{\text{tot}} &= \sum_i m_i \vec{r}_i \times \ddot{\vec{r}}_i = \sum_i m_i(\vec{r}_i - \vec{R}) \times \ddot{\vec{r}}_i + \sum_i m_i \vec{R} \times \ddot{\vec{r}}_i \\ &= \sum_i (\vec{r}_i - \vec{R}) \times \vec{F}_i + \vec{R} \times \vec{F}\end{aligned}$$

The former term is called the *torque* acting on the system about the centre of mass in the stationary frame. If we split $\vec{F}_i$ as $\vec{G}_i + \vec{H}_i$, where $\vec{G}_i$ is towards the centre of mass (i. e. parallel to the vector $\vec{r}_i - \vec{R}$) and $\vec{H}_i$ is orthogonal to it; then clearly the torque is given by $\sum_i (\vec{r}_i - \vec{R}) \times \vec{H}_i$. As before we can apply $g(t)^{-1}$ to it to obtain $\vec{\tau} = \sum_i \vec{u}_i \times \vec{h}_i$ as the torque acting on the top in the body frame; here $\vec{h}_i = g^{-1}\vec{H}_i$. On the other hand, using the above expression for $\ddot{\vec{r}}_i$ we see that

$$\begin{aligned}\dot{\vec{\lambda}} &= \sum m_i(\vec{r}_i - \vec{R}) \times (\ddot{\vec{r}}_i - \ddot{\vec{R}}) = \sum m_i(\vec{r}_i - \vec{R}) \times \ddot{\vec{r}}_i \\ &= \sum m_i(\vec{r}_i - \vec{R}) \times \Big(\vec{\omega} \times (\vec{\omega} \times (\vec{r}_i - \vec{R})) + \dot{\vec{\omega}} \times (\vec{r}_i - \vec{R})\Big) \\ &= \vec{\omega} \times \vec{\lambda} + \sum m_i(\vec{r}_i - \vec{R}) \times (\dot{\vec{\omega}} \times (\vec{r}_i - \vec{R}))\end{aligned}$$

We apply g^{-1} to express everything in the body frame.

$$\vec{\tau} = \vec{\Omega} \times \vec{\Lambda} + \dot{\vec{\Lambda}}$$

This expression in the body frame for the derivative of the angular momentum in terms of the torque and the angular velocity is called Euler's equation.

To summarise, the motion of an asymmetric top can separated into two components, the motion of its centre of mass and the motion about the centre of mass. The centre of mass behaves exactly like a point of mass M with position $\vec{R}$ subject to a force $\vec{F}$ under Newton's equations. The motion about the centre of mass is described in the body frame by a path g is the group of rotations, the moment of inertia tensor I and torque τ to which the body is subjected. The equation of motion in this case is Euler's equation. Since the motion of a point mass in a Newtonian system is well understood, we will concentrate on the latter.

The only expression that depends on the distribution of masses is I. Replacing the summation by integration and the masses by a mass density ρ, we see that another expression for the moment of inertia is

$$I(\vec{\Omega}) = \iiint \rho\ \vec{r} \times (\vec{\Omega} \times \vec{r})\ dx\,dy\,dz$$

Similarly, we see that

$$I_{\vec{w}} = \iiint \rho\ d(\vec{r}, \mathbb{R}\vec{w})^2\ dx\,dx\,dz$$

where $d(\vec{r}, \mathbb{R}\vec{w})$ denotes the distance of $\vec{r}$ from the line $\mathbb{R}\vec{w}$. If we perform these computations for a ellipsoid with principal axes being the co-ordinate axes with principal lengths a, b and c, then

$$(x, y, z).I((x, y, z)) = \text{const.} \cdot ((b^2 + c^2)x^2(c^2 + a^2)y^2(a^2 + b^2)z^2)$$

In particular, note that the "shape" of the surface $\vec{w}.I(\vec{w}) = 1$ is quite different from that of the boundary of the given ellipsoid.

We say that the rotation is *inertial* if the torque vanishes; i. e. $\vec{\tau} = 0$. It follows that $\dot{\vec{\lambda}} = 0$ so that the angular momentum $\vec{\lambda}$ in the stationary frame is conserved. In particular, the magnitude $||\vec{\Lambda}|| = ||\vec{\lambda}||$ of the angular momentum in the body frame is preserved. Moreover, taking the derivative of the rotational kinetic energy $T = \frac{1}{2}\vec{\omega}.\vec{\lambda} = \frac{1}{2}\vec{\Omega}.\vec{\Lambda}$ we see (using the symmetry of I) that

$$\dot{\vec{\Omega}}.\vec{\Lambda} = \dot{\vec{\Omega}}.I(\vec{\Omega}) = (\tfrac{1}{2}\vec{\Omega}.I(\dot{\vec{\Omega}}) + \tfrac{1}{2}\dot{\vec{\Omega}}.I(\vec{\Omega})) = \dot{T} = \tfrac{1}{2}\dot{\vec{\omega}}.\vec{\lambda} = \tfrac{1}{2}\dot{\vec{\Omega}}.\vec{\Lambda}$$

Thus $\dot{\vec{\Omega}}.\vec{\Lambda} = \dot{T} = 0$ and the so kinetic energy T in the body frame is preserved. It follows that $\vec{\Omega}$ lies on the intersection of the sphere S defined by $||\vec{\Omega}|| = \text{const.}$ and the ellipsoid E defined by $\vec{\Omega}.I(\vec{\Omega}) = \text{const.}$; the ellipsoid could be degenerate in case I is not invertible (but this won't happen for "solid" tops).

Poinsot offered a more precise description as follows. Consider the image $g \cdot E$ of the ellipsoid. The vector $\vec{\omega} = g \cdot \vec{\Omega}$ lies on this ellipsoid. Moreover, a vector $\vec{w}$ is tangent to E at $\vec{\Omega}$ if $\vec{w}.\vec{\Lambda} = \vec{w}.I(\vec{\Omega}) = 0$. Thus a vector $\vec{x}$ is tangent to $g \cdot E$ at $\vec{\omega}$ if $\vec{x}.\vec{\lambda} = 0$. Said differently, the tangent plane π to $g \cdot E$ at $\vec{\omega}$ consists of vectors $\vec{y}$ such that $\vec{y}.\vec{\lambda} = \vec{\omega}.\vec{\lambda}$. We recognise the latter expression as $2T$, twice the rotational kinetic energy, which is a constant of motion. In other words, π is also a constant of motion.

To summarise, the rotational motion g is such that the moving ellipsoid $g \cdot E$ remains tangent to a fixed plane and the point of tangency provides the axis of rotation; such a motion of the ellipsoid E is called rolling without slipping on the plane π. To recover the rotational motion of the original top we note that $(\vec{r}_i - \vec{R}) = g \cdot u_i$; so the top is "affixed" to the ellipsoid through its centre of mass with the body frame aligned so that the eigen-basis of the moment of inertia are the principal axes of the ellipsoid. *Warning*: the reader should beware that we are *not* describing the inertial rotational motion of a top shaped like E—rather the motion of E described gives a nice geometric description of the motion of the original top.